REAL NON-ABELIAN MIXED HODGE STRUCTURES FOR SCHEMATIC HOMOTOPY TYPES OF QUASI-PROJECTIVE VARIETIES

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Abstract. The relative Malcev homotopy type of a quasi-projective variety carries a canonical non-positively weighted algebraic mixed twistor structure (MTS), provided we restrict to extensions of local systems with trivial monodromy around the components of the divisor. This can be enriched to an analytic mixed Hodge structure (MHS), which becomes algebraic if we restrict to extensions of local systems underlying VHS.

We then show that every non-positively weighted MHS or MTS on homotopy types admits a canonical splitting over SL_2. For smooth varieties, this allows us to characterise the MHS or MTS in terms of the Gysin spectral sequence, together with the monodromy action at the Archimedean place. It also means that the relative Malcev homotopy groups carry canonical MTS or MHS.

Introduction

The main aims of this paper are to construct mixed Hodge structures on the real relative Malcev homotopy types of open complex varieties, and to investigate how far these can be recovered from the structures on cohomology groups of local systems, and in particular the Gysin spectral sequence. In this respect, this paper is a sequel to [Pri4], which considers the same question for proper complex varieties.

In [Mor], Morgan established the existence of natural mixed Hodge structures on the minimal model of the rational homotopy type of a smooth variety X, and used this to define natural mixed Hodge structures on the rational homotopy groups $\pi_*(X \otimes \mathbb{Q})$ of X. This construction was extended to singular varieties by Hain in [Hai].

For non-nilpotent topological spaces, the rational homotopy type is too crude an invariant to recover much information, so schematic homotopy types were introduced in [Toë], based on ideas from [Gro]. [Pri2] showed how to recover the groups $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R}$ from schematic homotopy types for very general topological spaces, and also introduced the intermediate notion of relative Malcev homotopy type, simultaneously generalising both rational and schematic homotopy types.

In [Pri4], the notions of mixed Hodge and mixed twistor structures on real relative Malcev homotopy types were introduced, and were constructed for homotopy types of compact Kähler manifolds. [Pri4] also introduced an important class of MHS or MTS — those which are SL_2-split or $\mathcal{S}$-split. These split on tensoring with the ring $\mathcal{S} := \mathbb{R}[x]$ whose Hodge filtration on $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$ is given by powers of $(x - i)$. It was then shown in [Pri4] that any $\mathcal{S}$-split MHS or MTS on relative Malcev homotopy types gives rise to MHS or MTS on the relative Malcev homotopy groups, with the latter also being $\mathcal{S}$-split. Adapting [DGMS] gave rise to an $\mathcal{S}$-splitting of the MHS and MTS for homotopy types of compact Kähler manifolds, and hence $\mathcal{S}$-split MHS/MTS on the homotopy groups.

This paper is broken into two main parts: we first adapt [Pri4] to construct MHS/MTS for relative Malcev homotopy types of quasi-projective varieties in §2, but only when the monodromy around the divisor is trivial. A more general case (unitary monodromy around

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the divisor) is addressed in §3. Whereas the $\mathcal{S}$-splittings of [Pri4] were realised concretely using the principle of two types, the second part of the paper (§§4–5) establishes abstract existence results for $\mathcal{S}$-splittings of general mixed Hodge and mixed twistor structures. These latter results are then needed to construct mixed Hodge and mixed twistor structures on relative Malcev homotopy groups of quasi-projective varieties (§5.4).

The structure of the paper is as follows: §1 recalls several basic results from [Pri4] concerning non-abelian filtrations, mixed Hodge structures and mixed twistor structures. These are adapted slightly here to specialise to non-positively weighted homotopy types.

§2 deals with the Malcev homotopy type $(Y,y)^{\rho,\mathrm{Mal}}$ of a quasi-projective variety $Y = X - D$ with respect to a Zariski-dense representation $\rho\colon \pi_1(X,y) \to R(\mathbb{R})$. For the local system $\mathbb{O}(R)$ on X corresponding to the regular representation $O(R)$, the construction of MHS and MTS is based on the complex $A^\bullet(X,\mathbb{O}(R))[\![D]\!]$, defined by modifying the $\mathbb{O}(R)$-valued de Rham complex by allowing logarithmic singularities around the divisor.

When Y is smooth, Theorem 2.21 establishes a non-positively weighted MTS on $(Y,y)^{\rho,\mathrm{Mal}}$, with the associated graded object $\mathrm{gr}^W(Y,y)^{\rho,\mathrm{Mal}}$ corresponding to the R-equivariant DGA

$$(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* j^{-1}\mathbb{O}(R))[-a], d_2),$$

where $d_2\colon \mathrm{H}^{a-b}(X,\mathbf{R}^b j_* j^{-1}\mathbb{O}(R)) \to \mathrm{H}^{a-b+2}(X,\mathbf{R}^{b-1} j_* j^{-1}\mathbb{O}(R))$ is the differential on the E_2 sheet of the Leray spectral sequence for $j : Y \to X$, and $\mathrm{H}^{a-b}(X,\mathbf{R}^b j_* j^{-1}\mathbb{O}(R))$ has weight $a+b$. Theorem 2.22 shows that if R-representations underlie variations of Hodge structure, then the MTS above extends to a non-positively weighted MHS on $(Y,y)^{\rho,\mathrm{Mal}}$. Theorem 2.30 gives the corresponding results for singular quasi-projective varieties Y, with $\mathrm{gr}^W(Y,y)^{\rho,\mathrm{Mal}}$ now characterised in terms of cohomology of a smooth simplicial resolution of Y.

In §3, these results are extended to Zariski-dense representations $\rho\colon \pi_1(Y,y) \to R(\mathbb{R})$ with unitary monodromy around local components of the divisor. The construction of MHS and MTS in these cases is much trickier than for trivial monodromy. The idea behind Theorem 3.16, inspired by [Mor], is to construct the Hodge filtration on the complexified homotopy type, and then to use homotopy limits of diagrams to glue this to the real form. When R-representations underlie variations of Hodge structure on Y, this gives a non-positively weighted MHS on $(Y,y)^{\rho,\mathrm{Mal}}$, with $\mathrm{gr}^W(Y,y)^{\rho,\mathrm{Mal}}$ corresponding to the R-equivariant DGA

$$(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2),$$

regarded as a Hodge structure via the VHS structure on $\mathbb{O}(R)$. For more general R, Theorem 3.19 gives a non-positively weighted MTS, with the construction based on homotopy gluing over an affine cover of the analytic space $\mathbb{P}^1(\mathbb{C})$. Simplicial resolutions then extend these results to singular varieties in Theorems 3.21 and Theorem 3.22. §3.5 discusses possible extensions to more general monodromy.

§4 is concerned with splittings of MHS and MTS on finite-dimensional vector spaces. Every mixed Hodge structure V splits on tensoring with the ring $\mathcal{S}$ defined above, giving an $\mathcal{S}$-linear isomorphism $V \otimes \mathcal{S} \cong (\mathrm{gr}^W V) \otimes \mathcal{S}$ preserving the Hodge filtration F. Differentiating with respect to V, this gives a map $\beta\colon (\mathrm{gr}^W V) \to (\mathrm{gr}^W V) \otimes \Omega(\mathcal{S}/\mathbb{R})$ from which V can be recovered. Theorem 4.8 shows that the $\mathcal{S}$-splitting can be chosen canonically, corresponding to imposing certain restrictions on β, and this gives an equivalence of categories. In Remark 4.11, β is explicitly related to the complex splitting of [Del4]. Theorem 4.15 then gives the corresponding results for mixed twistor structures.

The main result in §5 is Theorem 5.16, which shows that every non-positively weighted MHS or MTS on a real relative Malcev homotopy type admits a strictification, in the sense that it is represented by an R-equivariant DGA in ind-MHS or ind-MTS. Corollary 5.22

then applies the results of §4 to give canonical $\mathcal{S}$-splittings for such MHS or MTS, while Corollary 5.23 shows that the splittings give equivalences $(Y,y)^{\rho,\mathrm{Mal}} \simeq \mathrm{gr}^W (Y,y)^{\rho,\mathrm{Mal}}$. Corollary 5.24 shows that they give rise to MHS or MTS on homotopy groups, and this is applied to quasi-projective varieties in Corollary 5.35. There are various consequences for deformations of representations (Proposition 5.33). Finally, Theorem 5.38 shows that for projective varieties, the canonical $\mathcal{S}$-splittings coincide with the explicit $\mathcal{S}$-splittings established in [Pri4].

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Notation. For any affine scheme Y, write $O(Y) := \Gamma(Y, \mathscr{O}_Y)$.

Contents

1. Non-abelian filtrations

In this section, we summarise several results from [Pri4] concerning non-abelian generalisations of real mixed Hodge and mixed twistor structures.

Lemma 1.1. *There is an equivalence of categories between flat quasi-coherent $\mathbb{G}_m$-equivariant sheaves on $\mathbb{A}^1$, and exhaustive (i.e. $V = \bigcup_n F_n V$) filtered vector spaces, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ via the standard embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$.*

Proof. This is [Pri4, Lemma 1.6]. Given a filtered vector space V, the equivalence sets M to be the Rees module $\xi(V,F) := \bigoplus F_nV$, with $\mathbb{G}_m$-action given by setting F_nV to be weight n, and the $k[t]$-module structure determined by letting t be the inclusion $F_nV \hookrightarrow F_{n+1}V$. □

1.1. **Mixed Hodge and mixed twistor structures.**

Definition 1.2. Define C to be the real affine scheme $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{A}^1$ obtained from $\mathbb{A}^1_{\mathbb{C}}$ by restriction of scalars, so for any real algebra A, $C(A) = \mathbb{A}^1_{\mathbb{C}}(A \otimes_{\mathbb{R}} \mathbb{C}) \cong A \otimes_{\mathbb{R}} \mathbb{C}$. Choosing $i \in \mathbb{C}$ gives an isomorphism $C \cong \mathbb{A}^2_{\mathbb{R}}$, and we let C^* be the quasi-affine scheme $C - \{0\}$.

Define S to be the real algebraic group $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ obtained as in [Del1, 2.1.2] from $\mathbb{G}_{m,\mathbb{C}}$ by restriction of scalars. Note that there is a canonical inclusion $\mathbb{G}_m \hookrightarrow S$, and that S acts on C and C^* by inverse multiplication, i.e.

$$\begin{array}{ccc} S \times C & \to & C \\ (\lambda, w) & \mapsto & (\lambda^{-1}w). \end{array}$$

Remark 1.3. Fix an isomorphism $C \cong \mathbb{A}^2$, with co-ordinates u, v on C so that the isomorphism $C(\mathbb{R}) \cong \mathbb{C}$ is given by $(u,v) \mapsto u + iv$. Thus the algebra $O(C)$ associated to C is the polynomial ring $\mathbb{R}[u,v]$. S is isomorphic to the scheme $\mathbb{A}^2_{\mathbb{R}} - \{(u,v) : u^2 + v^2 = 0\}$. On $C_{\mathbb{C}}$, we have alternative co-ordinates $w = u + iv$ and $\bar{w} = u - iv$, which give the standard isomorphism $S_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Note that on C the co-ordinates w and $\bar{w}$ are of types $(-1,0)$ and $(0,-1)$ respectively.

Definition 1.4. Given an S-representation V, the inclusion $\mathbb{G}_m \hookrightarrow S$ (given by $v = 0$ in the co-ordinates above) gives a grading on V, which we denote by

$$V = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n V.$$

Equivalently, $\mathcal{W}_n(V \otimes \mathbb{C})$ is the sum of elements of type (p,q) for $p + q = n$.

1.1.1. *Mixed Hodge structures.*

Lemma 1.5. *The category of flat S-equivariant quasi-coherent sheaves on C^* is equivalent to the category of pairs (V,F), where V is a real vector space and F an exhaustive decreasing filtration on $V \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof. This is contained in [Pri4, Corollary 1.8]. The construction is given by first forming the complex Rees module

$$\xi(V_{\mathbb{C}}, F, \bar{F}) := \bigoplus_{p,q \in \mathbb{Z}} w^{-p}\bar{w}^{-q}(F^pV_{\mathbb{C}}) \cap (\bar{F}^qV_{\mathbb{C}})$$

with respect to F and $\bar{F}$. This has a $\mathbb{C}[w, \bar{w}$-module structure, and an S-action as a submodule of $V_{\mathbb{C}}[w, w^{-1}, \bar{w}, \bar{w}^{-1}]$. We then form $\xi(V, \mathbb{F}) \subset \xi(V_{\mathbb{C}}, F, \bar{F})$ to consist of real elements. This a flat S-equivariant $\mathbb{R}[u,v]$-module, so defines a flat S-equivariant quasi-coherent sheaf on C, and hence pulls back to one on C^*. □

Definition 1.6. Given an affine scheme X over $\mathbb{R}$, we define an algebraic mixed Hodge structure X_{MHS} on X to consist of the following data:

1. an $\mathbb{G}_m \times S$-equivariant affine morphism $X_{\mathrm{MHS}} \to \mathbb{A}^1 \times C^*$,
2. a real affine scheme $\underline{\mathrm{gr}}X_{\mathrm{MHS}}$ equipped with an S-action,
3. an isomorphism $X \cong X_{\mathrm{MHS}} \times_{(\mathbb{A}^1 \times C^*),(1,1)} \operatorname{Spec} \mathbb{R}$,
4. a $\mathbb{G}_m \times S$-equivariant isomorphism $\underline{\mathrm{gr}}X_{\mathrm{MHS}} \times C^* \cong X_{\mathrm{MHS}} \times_{\mathbb{A}^1,0} \operatorname{Spec} \mathbb{R}$, where $\mathbb{G}_m$ acts on $\underline{\mathrm{gr}}X_{\mathrm{MHS}}$ via the inclusion $\mathbb{G}_m \hookrightarrow S$. This is called the opposedness isomorphism.

Definition 1.7. Define a (real) quasi-MHS to be a real vector space V, equipped with an exhaustive increasing filtration W on V, and an exhaustive decreasing filtration F on $V \otimes \mathbb{C}$.

We adopt the convention that a (real) MHS is a finite-dimensional quasi-MHS on which W is Hausdorff, satisfying the opposedness condition

$$\mathrm{gr}^W_n \mathrm{gr}^i_F \mathrm{gr}^j_{\bar{F}}(V \otimes \mathbb{C}) = 0$$

for $i + j \neq n$.

Define a (real) ind-MHS to be a filtered direct limit of MHS. Say that an ind-MHS is bounded below if $W_N V = 0$ for $N \ll 0$.

Lemma 1.8. *The category of flat $\mathbb{G}_m \times S$-equivariant quasi-coherent sheaves M on $\mathbb{A}^1 \times C^*$ is equivalent to the category of quasi-MHS.*

Under this equivalence, bounded below ind-MHS (V, W, F) correspond to flat algebraic mixed Hodge structures M on V whose weights with respect to the $\mathbb{G}_m \times 1$-action are bounded below.

A real splitting of the Hodge filtration is equivalent to giving a (real) Hodge structure on V (i.e. an S-action).

Proof. This is [Pri4, Proposition 1.40]. The construction is given by combining Lemmas 1.1 and 1.5. □

1.1.2. *Mixed twistor structures.*

Definition 1.9. Adapting [Sim1] §1 from complex to real structures, say that a twistor structure on a real vector space V consists of a vector bundle $\mathscr{E}$ on $\mathbb{P}^1_{\mathbb{R}}$, with an isomorphism $V \cong \mathscr{E}_1$, the fibre of $\mathscr{E}$ over $1 \in \mathbb{P}^1$.

Lemma 1.10. *The category of finite flat algebraic twistor filtrations on real vector spaces is equivalent to the category of twistor structures.*

Proof. This is [Pri4, Proposition 1.8]. The flat algebraic twistor filtration is a flat $\mathbb{G}_m$-equivariant quasi-coherent sheaf M on C^*, with $M|_1 = V$. Taking the quotient by the right $\mathbb{G}_m$-action, M corresponds to a flat quasi-coherent sheaf $M_{\mathbb{G}_m}$ on $[C^*/\mathbb{G}_m]$. Now, $[C^*/\mathbb{G}_m] \cong [(\mathbb{A}^2 - \{0\})/\mathbb{G}_m] = \mathbb{P}^1$, so Lemma 1.1 implies that $M_{\mathbb{G}_m}$ corresponds to a flat quasi-coherent sheaf $\mathscr{E}$ on $\mathbb{P}^1$. Note that $\mathscr{E}_1 = (M|_{\mathbb{G}_m})_{\mathbb{G}_m} \cong M_1 \cong V$, as required. □

Definition 1.11. Let $\widetilde{C^*} \to C^*$ be the étale covering of C^* given by cutting out the divisor $\{u - iv = 0\}$ from $C^* \otimes_{\mathbb{R}} \mathbb{C}$, for co-ordinates u, v as in Definition 1.3.

Note that $\widetilde{C^*} \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, with the isomorphism given by sending (u, v) to $(u + iv, u - iv)$.

Definition 1.12. Adapting [Sim1] §1 from complex to real structures, say that a (real) mixed twistor structure (real MTS) on a real vector space V consists of a finite locally free sheaf $\mathscr{E}$ on $\mathbb{P}^1_{\mathbb{R}}$, equipped with an exhaustive Hausdorff increasing filtration by locally free subsheaves $W_i\mathscr{E}$, such that for all i the graded bundle $\mathrm{gr}^W_i \mathscr{E}$ is semistable of slope i (i.e. a direct sum of copies of $\mathscr{O}_{\mathbb{P}^1}(i)$). We also require an isomorphism $V \cong \mathscr{E}_1$, the fibre of $\mathscr{E}$ over $1 \in \mathbb{P}^1$.

Define a quasi-MTS on V to be a flat quasi-coherent sheaf $\mathscr{E}$ on $\mathbb{P}^1_{\mathbb{R}}$, equipped with an exhaustive increasing filtration by quasi-coherent subsheaves $W_i\mathscr{E}$, together with an isomorphism $V \cong \mathscr{E}_1$. Define an ind-MTS to be a filtered direct limit of real MTS, and say that an ind-MTS $\mathscr{E}$ on V is bounded below if $W_N\mathscr{E} = 0$ for $N \ll 0$.

Definition 1.13. Given an affine scheme X over $\mathbb{R}$, we define an algebraic mixed twistor structure X_{MTS} on X to consist of the following data:

(1) an $\mathbb{G}_m \times \mathbb{G}_m$-equivariant affine morphism $X_{\mathrm{MTS}} \to \mathbb{A}^1 \times C^*$,

(2) a real affine scheme $\underline{\mathrm{gr}}X_{\mathrm{MTS}}$ equipped with a $\mathbb{G}_m$-action,
(3) an isomorphism $X \cong X_{\mathrm{MTS}} \times_{(\mathbb{A}^1 \times C^*),(1,1)} \operatorname{Spec} \mathbb{R}$,
(4) a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant isomorphism $\underline{\mathrm{gr}}X_{\mathrm{MTS}} \times C^* \cong X_{\mathrm{MTS}} \times_{\mathbb{A}^1,0} \operatorname{Spec} \mathbb{R}$. This is called the opposedness isomorphism.

Lemma 1.14. *The category of flat $\mathbb{G}_m \times \mathbb{G}_m$-equivariant quasi-coherent sheaves on $\mathbb{A}^1 \times C^*$ is equivalent to the category of quasi-MTS.*

Under this equivalence, bounded below ind-MTS on V correspond to flat algebraic mixed twistor structures $\xi(V, \mathrm{MTS})$ on V whose weights with respect to the $\mathbb{G}_m \times 1$-action are bounded below.

Proof. This is [Pri4, Proposition 1.48]. The construction is given by combining Lemmas 1.1 and 1.10. □

1.2. **Mixed Hodge and mixed twistor structures on Malcev homotopy types.**

1.2.1. *Relative Malcev homotopy types.*

Definition 1.15. Given a reductive real pro-algebraic monoid M, let $DG_{\mathbb{Z}}\mathrm{Alg}(M)$ (resp. $DG\mathrm{Alg}(M)$) be the category of R-representations in $\mathbb{Z}$-graded cochain graded-commutative $\mathbb{R}$-algebras (resp. non-negatively graded cochain graded-commutative $\mathbb{R}$-algebras).

For an M-representation A in algebras, we define $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$ (resp. $DG\mathrm{Alg}_A(M)$) to be the comma category $A{\downarrow}DG_{\mathbb{Z}}\mathrm{Alg}(M)$ (resp. $A{\downarrow}DG\mathrm{Alg}(M)$).

Denote the opposite categories by $dg_{\mathbb{Z}}\mathrm{Aff}_A(M)$ and $dg\mathrm{Aff}_A(M)$. Given an object $A \in DG_{\mathbb{Z}}\mathrm{Alg}(M)_*$, write $\operatorname{Spec} A \in dg_{\mathbb{Z}}\mathrm{Aff}(M)_*$ for the corresponding object of the opposite category. For each of the categories $\mathcal{C}$ above, let $\mathrm{Ho}(\mathcal{C})$ be the category obtained by formally inverting quasi-isomorphisms.

Definition 1.16. Given a reductive pro-algebraic monoid M, and an M-representation Y in schemes, define $DG_{\mathbb{Z}}\mathrm{Alg}_Y(M)$ to be the category of M-equivariant quasi-coherent $\mathbb{Z}$-graded graded-commutative cochain algebras on Y. Define a weak equivalence in this category to be a map giving isomorphisms on cohomology sheaves (over Y), and define $\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_Y(M))$ to be the homotopy category obtained by localising at weak equivalences. Define the categories $dg_{\mathbb{Z}}\mathrm{Aff}_Y(M), \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_Y(M))$ to be the respective opposite categories.

When Y is affine, define $DG\mathrm{Alg}_Y(M) \subset DG_{\mathbb{Z}}\mathrm{Alg}_Y(M)$ to consist of non-negatively graded cochain graded-commutative $\mathbb{R}$-algebras on Y, with $dg\mathrm{Aff}_Y(M)$, $\mathrm{Ho}(DG\mathrm{Alg}_Y(M))$ and $Ho(dg_{\mathbb{Z}}\mathrm{Aff}_Y(M))$ defined similarly.

Definition 1.17. Given a reductive pro-algebraic monoid K acting on a reductive pro-algebraic monoid M and on a scheme Y, define $dg_{\mathbb{Z}}\mathrm{Aff}_Y(M)_*(K)$ to be the category $(Y \times M) \downarrow dg_{\mathbb{Z}}\mathrm{Aff}_Y(M \rtimes K)$ of objects under $M \times Y$. Note that this is not the same as $dg_{\mathbb{Z}}\mathrm{Aff}_Y(M \rtimes K)_* = (Y \times M \rtimes K) \downarrow dg_{\mathbb{Z}}\mathrm{Aff}_Y(M \rtimes K)$. When Y is affine, define $dg\mathrm{Aff}_Y(M)_*(K)$ similarly.

Definition 1.18. Recall from [Pri4, Proposition 3.34] that for a reductive pro-algebraic group R, the relative Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$ of a pointed manifold (X,x) relative to $\rho : \pi_1(X,x) \to R(\mathbb{R})$ is given in $dg\mathrm{Aff}(R)_*$ by $R \xrightarrow{\operatorname{Spec} x^*} \operatorname{Spec} A^\bullet(X, \mathbb{O}(R))$, where $\mathbb{O}(R)$ is the local system on X corresponding to the left action of $\pi_1(X,x)$ on $O(R)$.

1.2.2. *Hodge and twistor structures.*

Definition 1.19. Define the real algebraic group S^1 to be the circle group, whose A-valued points are given by $\{(a,b) \in A^2 \,:\, a^2 + b^2 = 1\}$. Note that $S^1 \hookrightarrow S$, and that $S/\mathbb{G}_m \cong S^1$. This latter S-action gives S^1 a split Hodge filtration.

The following definitions and results are taken from [Pri4, §4]. Fix a real reductive pro-algebraic group R, a pointed connected topological space (X,x), and a Zariski-dense morphism $\rho : \pi_1(X,x) \to R(\mathbb{R})$.

Definition 1.20. An algebraic Hodge filtration on a pointed Malcev homotopy type $(X,x)^{\rho,\mathrm{Mal}}$ consists of the following data:

1. an algebraic action of S^1 on R,
2. an object $(X,x)^{\rho,\mathrm{Mal}}_{\mathbb{F}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(S))$, where the S-action on R is defined via the isomorphism $S/\mathbb{G}_m \cong S^1$, while the $R \rtimes S$-action on R combines multiplication by R with conjugation by S.
3. an isomorphism $(X,x)^{\rho,\mathrm{Mal}} \cong (X,x)_{\mathbb{F}} \times^{\mathbf{R}}_{C^*,1} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*)$.

Definition 1.21. An algebraic twistor filtration on a pointed Malcev homotopy type $(X,x)^{\rho,\mathrm{Mal}}$ consists of the following data:

1. an object $(X,x)^{\rho,\mathrm{Mal}}_{\mathbb{T}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(\mathbb{G}_m))$,
2. an isomorphism $(X,x)^{\rho,\mathrm{Mal}} \cong (X,x)^{\rho,\mathrm{Mal}}_{\mathbb{T}} \times^{\mathbf{R}}_{C^*,1} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*)$.

Definition 1.22. Mat_n is the algebraic monoid of $n \times n$-matrices. Thus $\mathrm{Mat}_1 \cong \mathbb{A}^1$, so acts on $\mathbb{A}^1$ by multiplication. Note that the inclusion $\mathbb{G}_m \hookrightarrow \mathrm{Mat}_1$ identifies Mat_1-representations with non-negatively weighted $\mathbb{G}_m$-representations.

Let $\bar{S} := (\mathrm{Mat}_1 \times S^1)/(-1,-1)$, giving a real algebraic monoid whose subgroup of units is S, via the isomorphism $S \cong (\mathbb{G}_m \times S^1)/(-1,-1)$. There is thus a morphism $\bar{S} \to S^1$ given by $(m,u) \mapsto u^2$, extending the isomorphism $S/\mathbb{G}_m \cong S^1$.

Note that $\bar{S}$-representations correspond via the morphism $S \to \bar{S}$ to real Hodge structures of non-negative weights. In the co-ordinates of Remark 1.3,

$$\bar{S} = \mathrm{Spec}\,\mathbb{R}[u, v, \frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2}].$$

The following adapts [Pri4, Definition 4.4] to non-positive weights, replacing $\mathbb{G}_m$ and S with Mat_1 and $\bar{S}$ respectively.

Definition 1.23. A non-positively weighted algebraic mixed Hodge structure $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ on a pointed Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$ consists of the following data:

1. an algebraic action of S^1 on R,
2. an object
$$(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times S)),$$
where S acts on R via the S^1-action, using the canonical isomorphism $S^1 \cong S/\mathbb{G}_m$,
3. an object
$$\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*(\bar{S})),$$
4. an isomorphism $(X,x)^{R,\mathrm{Mal}} \cong (X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \times^{\mathbf{R}}_{(\mathbb{A}^1 \times C^*),(1,1)} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*)$,
5. an isomorphism (called the opposedness isomorphism)
$$\theta^\sharp(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}) \times C^* \cong (X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(\mathrm{Mat}_1 \times S)),$$
for the canonical map $\theta : \mathrm{Mat}_1 \times S \to \bar{S}$ given by combining the inclusion $\mathrm{Mat}_1 \hookrightarrow \bar{S}$ with the inclusion $S \hookrightarrow \bar{S}$.

Definition 1.24. An non-positively weighted algebraic mixed twistor structure $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ on a pointed Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$ consists of the following data:

1. an object
$$(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)),$$

(2) an object $\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*(\mathrm{Mat}_1))$,
(3) an isomorphism $(X,x)^{R,\mathrm{Mal}} \cong (X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{(\mathbb{A}^1\times C^*),(1,1)} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}(R)_*)$,
(4) an isomorphism (called the opposedness isomorphism)

$$\theta^\sharp(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \times C^* \cong (X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R} \in \mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)),$$

for the canonical map $\theta : \mathrm{Mat}_1 \times \mathbb{G}_m \to \mathrm{Mat}_1$ given by combining the identity on Mat_1 with the inclusion $\mathbb{G}_m \hookrightarrow \mathrm{Mat}_1$.

2. Algebraic MHS/MTS for quasi-projective varieties I

Fix a smooth compact Kähler manifold X, a divisor D locally of normal crossings, and set $Y := X - D$. Let $j : Y \to X$ be the inclusion morphism.

Definition 2.1. Denote the sheaf of real $\mathcal{C}^\infty$ n-forms on X by $\mathscr{A}^n_X$, and let $\mathscr{A}^\bullet_X$ be the resulting complex (the real sheaf de Rham complex on X).

Let $\mathscr{A}^\bullet_X[\![D]\!] \subset j_*\mathscr{A}_Y$ be the sheaf of dg $\mathscr{A}^\bullet_X$-subalgebras locally generated by $\{\log r_i, d\log r_i, d^c \log r_i\}_{1\le i\le m}$, where D is given in local co-ordinates by $D = \bigcup_{i=1}^m \{z_i = 0\}$, and $r_i = |z_i|$.

Let $\mathscr{A}^\bullet_X\langle D\rangle \subset j_*\mathscr{A}_Y \otimes \mathbb{C}$ be the sheaf of dg $\mathscr{A}^\bullet_X \otimes \mathbb{C}$-subalgebras locally generated by $\{d\log z_i\}_{1\le i\le m}$.

Note that $d^c \log r_i = d \arg z_i$.

Definition 2.2. Construct increasing filtrations on $\mathscr{A}^\bullet_X\langle D\rangle$ and $\mathscr{A}^\bullet_X[\![D]\!]$ by setting

$$\begin{aligned} J_0\mathscr{A}^\bullet_X[\![D]\!] &= \mathscr{A}^\bullet_X, \\ J_0\mathscr{A}^\bullet_X\langle D\rangle &= \mathscr{A}^\bullet_X \otimes \mathbb{C}, \end{aligned}$$

then forming $J_r\mathscr{A}^\bullet_X\langle D\rangle \subset \mathscr{A}^\bullet_X\langle D\rangle$ and $J_r\mathscr{A}^\bullet_X[\![D]\!] \subset \mathscr{A}^\bullet_X[\![D]\!]$ inductively by the local expressions

$$J_r\mathscr{A}_X\langle D\rangle = \sum_i J_{r-1}\mathscr{A}^\bullet_X\langle D\rangle d\log z_i,$$

$$J_r\mathscr{A}_X[\![D]\!] = \sum_i J_{r-1}\mathscr{A}^\bullet_X[\![D]\!] \log r_i + \sum_i J_{r-1}\mathscr{A}^\bullet_X[\![D]\!] d\log r_i + \sum_i J_{r-1}\mathscr{A}^\bullet_X[\![D]\!] d^c \log r_i,$$

for local co-ordinates as above.

Given any cochain complex V, we denote the good truncation filtration by $\tau_n V := \tau^{\le n} V$.

Lemma 2.3. *The maps*

$$\begin{aligned} (\mathscr{A}^\bullet_X\langle D\rangle, J) \leftarrow \quad & (\mathscr{A}^\bullet_X\langle D\rangle, \tau) \quad \to (j_*\mathscr{A}^\bullet_Y, \tau) \\ (\mathscr{A}^\bullet_X[\![D]\!], J) \leftarrow \quad & (\mathscr{A}^\bullet_X[\![D]\!], \tau) \quad \to (j_*\mathscr{A}^\bullet_Y, \tau) \end{aligned}$$

are filtered quasi-isomorphisms of complexes of sheaves on X.

Proof. This is essentially the same as [Del1] Prop 3.1.8, noting that the inclusion $\mathscr{A}^\bullet_X\langle D\rangle \hookrightarrow \mathscr{A}^\bullet_X[\![D]\!] \otimes \mathbb{C}$ is a filtered quasi-isomorphism, because $\mathscr{A}^\bullet_X[\![D]\!] \otimes \mathbb{C}$ is locally freely generated over $\mathscr{A}^\bullet_X\langle D\rangle$ by the elements $\log r_i$ and $d\log r_i$. □

An immediate consequence of this lemma is that for all $m \ge 0$, the flabby complex $\mathrm{gr}^J_m\mathscr{A}_X[\![D]\!]$ is quasi-isomorphic to $\mathbf{R}^m j_*\mathbb{R}$.

Definition 2.4. For any real local system $\mathbb{V}$ on X, define

$$\mathscr{A}^\bullet_X(\mathbb{V}) := \mathscr{A}^\bullet_X \otimes_{\mathbb{R}} \mathbb{V}, \quad \mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle := \mathscr{A}^\bullet_X\langle D\rangle \otimes_{\mathbb{R}} \mathbb{V}, \quad \mathscr{A}^\bullet_X(\mathbb{V})[\![D]\!] := \mathscr{A}^\bullet_X[\![D]\!] \otimes_{\mathbb{R}} \mathbb{V}.$$

$$A^\bullet(X,\mathbb{V}) := \Gamma(X, \mathscr{A}^\bullet_X(\mathbb{V})), \qquad A^\bullet(X,\mathbb{V})\langle D\rangle := \Gamma(X, \mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle),$$

$$A^\bullet(X,\mathbb{V})[\![D]\!] := \Gamma(X, \mathscr{A}^\bullet_X(\mathbb{V})[\![D]\!]).$$

These inherit filtrations, given by

$$J_rA^\bullet(X,\mathbb{V})\langle D\rangle := \Gamma(X, J_r\mathscr{A}^\bullet_X\langle D\rangle\otimes\mathbb{V}),$$
$$J_rA^\bullet(X,\mathbb{V})[\![D]\!] := \Gamma(X, J_r\mathscr{A}^\bullet_X[\![D]\!]\otimes\mathbb{V}).$$

Note that Lemma 2.3 implies that for all $m\geq 0$, the flabby complex $\mathrm{gr}^J_m\mathscr{A}_X(\mathbb{V})[\![D]\!]$ (resp. $\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle$) is quasi-isomorphic to $\mathbf{R}^m j_*(j^{-1}\mathbb{V})\cong\mathbb{V}\otimes\mathbf{R}^m j_*\mathbb{R}$ (resp. $\mathbf{R}^m j_*(j^{-1}\mathbb{V})\otimes\mathbb{C}$).

Remark 2.5. The filtration J essentially corresponds to the weight filtration W of [Del1, 3.1.5]. However, the true weight filtration on cohomology, and hence on homotopy types, is given by the décalage $\mathrm{Dec}\,J$ (as in [Del1, Theorem 3.2.5] or [Mor]). Since $\mathrm{Dec}\,J$ gives the correct notion of weights, not only for mixed Hodge structures but also for Frobenius eigenvalues in the ℓ-adic case of [Pri5], we reserve the terminology "weight filtration" for $W:=\mathrm{Dec}\,J$.

2.1. **The Hodge and twistor filtrations.** If we write J for the complex structure on $A^\bullet(X)$, then there is a differential $d^c:=J^{-1}dJ$ on the underlying graded algebra $A^*(X)$. Note that $dd^c+d^cd=0$.

Definition 2.6. There is an action of S on $A^*(X)$, which we will denote by $a\mapsto\lambda\diamond a$, for $\lambda\in\mathbb{C}^\times=S(\mathbb{R})$. For $a\in(A^*(X)\otimes\mathbb{C})^{pq}$, the action is given by

$$\lambda\diamond a:=\lambda^p\bar{\lambda}^q a.$$

It follows from [Sim2] Theorem 1 that there exists a harmonic metric on every semisimple real local system $\mathbb{V}$ on X. We then decompose the associated connection $D:\mathscr{A}^0_X(\mathbb{V})\to\mathscr{A}^1_X(\mathbb{V})$ as $D=d^++\vartheta$ into antisymmetric and symmetric parts, and let $D^c:=i\diamond d^+-i\diamond\vartheta$. Note that this decomposition is independent of the choice of metric, since the pluriharmonic metric is unique up to global automorphisms $\Gamma(X,\mathrm{Aut}(\mathbb{V}))$.

Definition 2.7. Given a semisimple real local system $\mathbb{V}$ on X, define the sheaves $\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})$ and $\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})[\![D]\!]$ of cochain complexes on $X_{\mathrm{an}}\times C_{\mathrm{Zar}}$ by

$$\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})=(\mathscr{A}^*_X(\mathbb{V})\otimes_{\mathbb{R}}O(C),uD+vD^c),$$
$$\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})[\![D]\!]=(\mathscr{A}^*_X(\mathbb{V})[\![D]\!]\otimes_{\mathbb{R}}O(C),uD+vD^c),$$

for co-ordinates u,v as in §1.1. We denote the differential by $\tilde{D}:=uD+vD^c$.

Define the quasi-coherent sheaf $\tilde{A}^\bullet(X,\mathbb{V})[\![D]\!]$ of cochain complexes on C by $\tilde{A}^\bullet(X,\mathbb{V})[\![D]\!]:=\mathrm{pr}_{C*}(\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})[\![D]\!])$.

Note that the $\diamond$ action on $\mathscr{A}$ gives an action of $\mathbb{G}_m\subset S$ on $\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})[\![D]\!]$ over C.

Definition 2.8. Given a semisimple local system $\mathbb{V}$ and an element $t\in S^1$, define the semisimple local system $t\circledast\mathbb{V}$ as follows. Decompose the connection $D:\mathscr{A}^0_X(\mathbb{V})\to\mathscr{A}^1_X(\mathbb{V})$ as $D=d^++\vartheta$ into antisymmetric and symmetric parts, and set

$$t\circledast D:=d^++t\diamond\vartheta=\partial+\bar{\partial}+t\theta+t^{-1}\bar{\theta},$$

then let $t\circledast\mathbb{V}:=\ker(t\circledast D:\mathscr{A}^0_X(\mathbb{V})\to\mathscr{A}^1_X(\mathbb{V}))$.

Definition 2.9. Assume that we have a semisimple local system $\mathbb{V}$, equipped with a discrete (resp. algebraic) action of S^1 on $\mathscr{A}^0_X(\mathbb{V})$ (denoted $v\mapsto t\circledast v$) such that

$$t\circledast(Dv)=(t\circledast D)(t\circledast v)$$

for $v\in\mathscr{A}^0_X(\mathbb{V})$ and $t\circledast D$ as above.

Then define a discrete $S(\mathbb{R})=\mathbb{C}^\times$-action (resp. an algebraic S-action) $\boxast$ on $\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})$ (and hence on $\tilde{\mathscr{A}}^\bullet_X(\mathbb{V})[\![D]\!]$) by

$$\lambda\boxast(a\otimes f\otimes v):=(\lambda\diamond a)\otimes\lambda(f)\otimes(\frac{\bar{\lambda}}{\lambda}\circledast v),$$

for $a \in \mathscr{A}_X$, $f \in O(C)$ and $v \in \mathbb{V}$. This gives an action on the global sections $\tilde{A}^\bullet_X(\mathbb{V})[\![D]\!]$ over C. Note that $\tilde{D}(\lambda \circledast b) = \lambda \circledast (\tilde{D}b)$, so this is indeed an action on cochain complexes.

In the above definition, observe that the discrete action of S^1 on $\mathscr{A}^0_X(\mathbb{V})$ is algebraic if and only if $\mathbb{V}$ is a weight 0 real variation of Hodge structure.

Definition 2.10. Given a Zariski-dense representation $\rho\colon \pi_1(X, jy) \to R(\mathbb{R})$, for R a pro-reductive pro-algebraic group, define an algebraic twistor filtration on the relative Malcev homotopy type $(Y, y)^{R,\mathrm{Mal}}$ by

$$(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}} := (R \times C^* \xrightarrow{\mathbf{Spec}\,(jy)^*} \mathbf{Spec}\,_{C^*} \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!]|_{C^*}),$$

in $\mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(\mathbb{G}_m))$, where $\mathbb{O}(R)$ is the local system of Definition 1.18, which is necessarily a sum of finite-dimensional semisimple local systems, and $\mathbb{G}_m \subset S$ acts via the $\circledast$ action of Definition 2.9.

A Zariski-dense representation $\rho\colon \pi_1(X, jy) \to R(\mathbb{R})$ is equivalent to a morphism $\varpi_1(X, jy)^{\mathrm{red}} \twoheadrightarrow R$ of pro-algebraic groups, where $\varpi_1(X, jy)^{\mathrm{red}}$ is the reductive quotient of the real pro-algebraic fundamental group $\varpi_1(X, jy)$. [Sim2] effectively gives a discrete S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$, corresponding (as in [Pri4, Lemma 5.7]) to the $\circledast$ action on semisimple local systems from Definition 2.8. This S^1-action thus descends to R if and only if $\mathbb{O}(R)$ satisfies the conditions of Definition 2.9. Moreover, the S^1-action is algebraic on R if and only if $\mathbb{O}(R)$ becomes a weight 0 variation of Hodge structures under the $\circledast$ action, by [Pri4, Proposition 5.12].

Definition 2.11. Take a Zariski-dense representation $\rho\colon \pi_1(X, jy) \to R(\mathbb{R})$, for R a pro-reductive pro-algebraic group to which the S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$ descends and acts algebraically. Then define an algebraic Hodge filtration on the relative Malcev homotopy type $(Y, y)^{R,\mathrm{Mal}}$ by

$$(Y, y)^{R,\mathrm{Mal}}_{\mathbb{F}} := (R \times C^* \xrightarrow{\mathbf{Spec}\,(jy)^*} \mathbf{Spec}\,_{C^*} \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!]|_{C^*}),$$

in $\mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(S))$, where the S-action is given by the $\circledast$ action of Definition 2.9.

If the S^1 action descends to R but is not algebraic, we still have the following:

Proposition 2.12. *The algebraic twistor filtration $(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}}$ of Definition 2.10 is equipped with a $(S^1)^\delta$-action (i.e. a discrete S^1-action) with the properties that*

(1) *the S^1-action and $\mathbb{G}_m$-actions commute,*
(2) *the projection $(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}} \to C^*$ is S^1-equivariant, and*
(3) *$-1 \in S^1$ acts as $-1 \in \mathbb{G}_m$.*

Proof. This is the same as the proof of [Pri4, Proposition 6.3]. The action comes from Definition 2.9, with $t \in (S^1)^\delta$ acting on $\mathscr{A}^*_X(\mathbb{O}(R))[\![D]\!]$ by $t \circledast (a \otimes v) = (t \diamond a) \otimes (t^2 \circledast v)$. $\square$

2.2. **Higher direct images and residues.**

Definition 2.13. Let $D^m \subset X$ denote the union of all m-fold intersections of local components of the divisor $D \subset X$, and set $D^{(m)}$ to be its normalisation. Write $\nu_m\colon D^{(m)} \to X$ for the composition of the normalisation map with the embedding of D^m, and set $C^{(m)} := \nu_m^{-1} D^{m+1}$.

As in [Tim2, 1.2], observe that $D^m - D^{m+1}$ is a smooth quasi-projective variety, isomorphic to $D^{(m)} - C^{(m)}$. Moreover, $D^{(m)}$ is a smooth projective variety, with $C^{(m)}$ a normal crossings divisor.

Definition 2.14. Recall from [Del1] Definition 2.1.13 that for $n \in \mathbb{Z}$, $\mathbb{Z}(n)$ is the lattice $(2\pi i)^n\mathbb{Z}$, equipped with the pure Hodge structure of type $(-n, -n)$. Given an abelian group A, write $A(n) := A \otimes_{\mathbb{Z}} \mathbb{Z}(n)$.

Definition 2.15. On $D^{(m)}$, define ε^m by the property that $\varepsilon^m(m)$ is the integral local system of orientations of D^m in X. Thus ε^n is the local system $\varepsilon^n_{\mathbb{Z}}$ defined in [Del1, 3.1.4].

Lemma 2.16. $\mathbf{R}^m j_*\mathbb{Z} \cong \nu_{m*}\varepsilon^m$.

Proof. This is [Del1, Proposition 3.1.9]. □

Lemma 2.17. *For any local system $\mathbb{V}$ on X, there is a canonical quasi-isomorphism*

$$\mathrm{Res}_m\colon \mathrm{gr}^J_m \mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \to \nu_{m*}\mathscr{A}^\bullet_{D^{(m)}}(\mathbb{V}\otimes_{\mathbb{R}} \varepsilon^m_{\mathbb{C}})[-m]$$

of cochain complexes on X.

Proof. We follow the construction of [Del1, 3.1.5.1]. In a neighbourhood where D is given locally by $\bigcup_i\{z_i = 0\}$, with $\omega \in \mathscr{A}^\bullet_X(\mathbb{V})$, we set

$$\mathrm{Res}_m(\omega \wedge d\log z_1 \wedge \ldots \wedge d\log z_m) := \omega|_{D^{(m)}} \otimes \epsilon(z_1,\ldots,z_m),$$

where $\epsilon(z_1,\ldots,z_m)$ denotes the orientation of the components $\{z_1 = 0\},\ldots,\{z_m = 0\}$.

That Res_m is a quasi-isomorphism follows immediately from Lemmas 2.3 and 2.16. □

2.3. **Opposedness.** Fix a Zariski-dense representation $\rho\colon \pi_1(X, jy) \to R(\mathbb{R})$, for R a pro-reductive pro-algebraic group.

Proposition 2.18. *If the S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$ descends to an algebraic action on R, then for the algebraic Hodge filtration $(Y, y)^{R,\mathrm{Mal}}_{\mathbb{H}}$ of Definition 2.11, the $R \rtimes S$-equivariant cohomology sheaf*

$$\mathscr{H}^a(\mathrm{gr}^J_b \mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{H}})$$

on C^* defines a pure ind-Hodge structure of weight $a+b$, corresponding to the $\circledast$ S-action on

$$\mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b).$$

Proof. We need to show that $\mathrm{H}^a(\mathrm{gr}^J_b \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!])|_{C^*}$ corresponds to a pure ind-Hodge structure of weight $a+b$, or equivalently a sum of vector bundles of slope $a+b$. We are therefore led to study the complex $\mathrm{gr}^J_b \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!])|_{C^*}$ on $X \times C^*$, since

$$\mathrm{H}^a(\mathrm{gr}^J_b \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!])|_{C^*} = \mathrm{H}^a(X, \mathrm{gr}^J_b \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!])|_{C^*}.$$

In a neighbourhood where D is given locally by $\bigcup_i\{z_i = 0\}$, $\mathrm{gr}^J \tilde{\mathscr{A}}^\bullet_X[\![D]\!]$ is the $\tilde{\mathscr{A}}^\bullet_X$-algebra generated by the classes $[\log|z_i|]$ $[d\log|z_i|]$ and $[d^c\log|z_i|]$ in gr^J_1. Let $\widetilde{C^*} \to C^*$ be the étale covering of Definition 1.11. Now, $\tilde{d} = ud + vd^c = (u+iv)\partial + (u-iv)\bar{\partial}$, so $\mathrm{gr}^J \tilde{\mathscr{A}}^\bullet_X[\![D]\!]|_{\widetilde{C^*}}$ is the $\mathrm{gr}^J \tilde{\mathscr{A}}^\bullet_X|_{\widetilde{C^*}}$-algebra generated by $[\log|z_i|], \tilde{d}[\log|z_i|], [d\log z]$.

Since $\tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!] = \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R)) \otimes_{\mathscr{A}^\bullet_X} \mathscr{A}^\bullet_X[\![D]\!]$, we have an S-equivariant quasi-isomorphism

$$\tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R)) \otimes_{\mathscr{A}^\bullet_X} \mathrm{gr}^J \mathscr{A}^\bullet_X\langle D\rangle|_{\widetilde{C^*}} \hookrightarrow \mathrm{gr}^J \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!]|_{\widetilde{C^*}},$$

as the right-hand side is generated over the left by $[\log|z_i|], \tilde{d}[\log|z_i|]$.

Now, Lemma 2.17 gives a quasi-isomorphism

$$\mathrm{Res}_b\colon \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R)) \otimes_{\mathscr{A}^\bullet_X} \mathrm{gr}^J_b \mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \to \nu_{b*}\tilde{\mathscr{A}}^\bullet_X \otimes_{\mathscr{A}^\bullet_X} \mathscr{A}^\bullet_{D^{(b)}}((\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b_{\mathbb{C}})[-b],$$

and the right-hand side is just

$$\nu_{b*}\tilde{\mathscr{A}}^\bullet_{D^{(b)}}(\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b_{\mathbb{C}})[-b].$$

Therefore

$$\mathrm{gr}^J_b \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!]|_{\widetilde{C^*}} \simeq \nu_{b*}\tilde{\mathscr{A}}^\bullet_{D^{(b)}}(\mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b_{\mathbb{C}})[-b]|_{\widetilde{C^*}},$$

and in particular Res_b defines an isomorphism

$$\mathrm{H}^a(\mathrm{gr}^J_b \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!]|_{\widetilde{C^*}}) \cong \mathrm{H}^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b_{\mathbb{C}})|_{\widetilde{C^*}}).$$

As in [Pri4, §1.1.2], we have an étale pushout $C^* = \widetilde{C^*} \cup_{S_{\mathbb{C}}} S$ of affine schemes, so to give an isomorphism $\mathscr{F} \to \mathscr{G}$ of quasi-coherent sheaves on C^* is the same as giving an isomorphism $f : \mathscr{F}|_{\widetilde{C^*}} \to \mathscr{G}|_{\widetilde{C^*}}$ such that $f|_{S_{Cx}}$ is real, in the sense that $f = \bar{f}$ on $S_{\mathbb{C}}$. Since $\tilde{d} \log |z_i| = (u + iv) d \log z_i + (u - iv) d \log \bar{z}_i$ is a boundary, we deduce that $[i(u - iv)^{-1} d \log z_i] \sim [-i(u + iv)^{-1} d \log \bar{z}_i]$, so

$$\overline{(u - iv)^b \mathrm{Res}_b} = (u - iv)^b \mathrm{Res}_b,$$

making use of the fact that ε^b already contains a factor of i^b (coming from $\mathbb{Z}(-b)$).

Therefore $(u - iv)^b \mathrm{Res}_b$ gives an isomorphism

$$\mathrm{H}^a(\mathrm{gr}^J_b \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!])|_{C^*} \cong \mathrm{H}^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b))|_{C^*}.$$

Now, $d \log z_i$ is of type $(1, 0)$, while ε^b is of type (b, b) and $(u - iv)$ is of type $(0, -1)$, so it follows that $(u - iv)^b \mathrm{Res}_b$ is of type $(0, 0)$, i.e. S-equivariant.

As in [Pri4, Theorem 5.14], inclusion of harmonic forms gives an S-equivariant isomorphism

$$\mathrm{H}^{a-b}(\tilde{A}^\bullet(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b))|_{C^*} \cong \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b) \otimes \mathscr{O}_{C^*},$$

which is a pure twistor structure of weight $(a - b) + 2b = a + b$. Therefore

$$\mathscr{H}^a(\mathrm{gr}^J_b \tilde{\mathscr{A}}^\bullet_X(\mathbb{O}(R))[\![D]\!]|_{C^*}) \cong \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b) \otimes \mathscr{O}_{C^*}$$

is pure of weight $a + b$, as required. □

Proposition 2.19. *For the algebraic twistor filtration $(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}}$ of Definition 2.10, the $R \times \mathbb{G}_m$-equivariant cohomology sheaf*

$$\mathscr{H}^a(\mathrm{gr}^J_b \mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}})$$

on C^* defines a pure ind-twistor structure of weight $a + b$, corresponding to the canonical $\mathbb{G}_m$-action on

$$\mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b).$$

Proof. The proof of Proposition 2.18 carries over, replacing S-equivariance with $\mathbb{G}_m$-equivariance, and Theorem 5.14 with Theorem 6.1. □

Proposition 2.20. *If the S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$ descends to R, then the associated discrete S^1-action of Proposition 2.12 on $\mathscr{H}^a(\mathrm{gr}^J_b \mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}})$ corresponds to the $\circledast$ action of $S^1 \subset S$ (see Definition 2.9) on*

$$\mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b).$$

Proof. The proof of Proposition 2.18 carries over, replacing S-equivariance with discrete S-equivariance. □

Theorem 2.21. *There is a canonical non-positively weighted mixed twistor structure $(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ on $(Y, y)^{R,\mathrm{Mal}}$, in the sense of Definition 1.24.*

Proof. On $\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}} = \tilde{A}^\bullet(X, \mathbb{O}(R))[\![D]\!]|_{C^*}$, we define the filtration $\mathrm{Dec}\, J$ by

$$(\mathrm{Dec}\, J)_r(\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}})^n = \{a \in J_{r-n}(\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}})^n \,:\, \tilde{D}a \in J_{r-n-1}(\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}})^{n+1}\}.$$

For the Rees algebra construction ξ of Lemma 1.1, we then set $\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)$ to be

$$\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} := \xi(\mathscr{O}(Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}}, \mathrm{Dec}\, J),$$

noting that this is flat and that $(X, x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times_{\mathbb{A}^1, 1} \mathrm{Spec}\, \mathbb{R} = (Y, y)^{R,\mathrm{Mal}}_{\mathbb{T}}$, so

$$(X, x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{(\mathbb{A}^1 \times C^*), (1,1)} \mathrm{Spec}\, \mathbb{R} \simeq (X, x)^{R,\mathrm{Mal}}.$$

We define $\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in dg_{\mathbb{Z}}\mathrm{Aff}(R)_*(\mathrm{Mat}_1)$ by

$$\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} = \mathrm{Spec}\,(\bigoplus_{a,b} \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b)[-a], d_1),$$

where $d_1\colon \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b) \to \mathrm{H}^{a-b+2}(D^{(b-1)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^{b-1})$ is the differential in the E_1 sheet of the spectral sequence associated to the filtration J. Combining Lemmas 2.16 and 2.17, it follows that this is the same as the differential $\mathrm{H}^{a-b}(X, \mathbf{R}^b j_* j^{-1}\mathbb{O}(R)) \to \mathrm{H}^{a-b+2}(X, \mathbf{R}^{b-1} j_* j^{-1}\mathbb{O}(R))$ in the E_2 sheet of the Leray spectral sequence for $j : Y \to X$. The augmentation $\bigoplus_{a,b} \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b) \to O(R)$ is just defined to be the unique ring homomorphism $\mathrm{H}^0(X, \mathbb{O}(R)) = \mathbb{R} \to O(R)$.

In order to show that this defines a mixed twistor structure, it only remains to establish opposedness. Since $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ is flat,

$$(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R} \simeq (X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R},$$

and properties of Rees modules mean that this is just given by

$$\mathbf{Spec}_{\,C^*}(\mathrm{gr}^{\mathrm{Dec}\,J} \mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}}) \in dg_{\mathbb{Z}}\mathrm{Aff}_{C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m),$$

where the Mat_1-action assigns $\mathrm{gr}^{\mathrm{Dec}\,J}_n$ the weight n.

By [Del1, Proposition 1.3.4], décalage has the formal property that the canonical map

$$\mathrm{gr}^{\mathrm{Dec}\,J}_n \mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}}) \to (\bigoplus_a \mathscr{H}^a(\mathrm{gr}^J_{n-a} \mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}})[-a], d_1)$$

is a quasi-isomorphism. Since the right-hand side is just

$$(\bigoplus_a \mathrm{H}^{2a-n}(D^{(n-a)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^{n-a})[-a], d_1) \otimes \mathscr{O}_{C^*}$$

by Proposition 2.19, we have a quasi-isomorphism

$$(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \times C^* \cong (X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R}.$$

That this is $(\mathrm{Mat}_1 \times \mathbb{G}_m)$-equivariant follows because $\mathrm{H}^{2a-n}(D^{(n-a)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^{n-a})$ is of weight $2a-n+2(n-a) = n$ for the $\mathbb{G}_m$ action, and of weight n for the Mat_1-action, being $\mathrm{gr}^{\mathrm{Dec}\,J}_n$. ☐

Theorem 2.22. *If the local system on X associated to any R-representation underlies a polarisable variation of Hodge structure, then there is a canonical non-positively weighted mixed Hodge structure $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ on $(Y,y)^{R,\mathrm{Mal}}$, in the sense of Definition 1.23.*

Proof. We adapt the proof of Theorem 2.21, replacing Proposition 2.19 with Proposition 2.18. The first condition is equivalent to saying that the S^1 action descends to R and is algebraic, by [Pri4, Proposition 5.12]. We therefore set

$$\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} := \xi(\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{H}}, \mathrm{Dec}\,J),$$

for $(Y,y)^{R,\mathrm{Mal}}_{\mathbb{H}}$ as in Definition 2.11, and let

$$\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} = \mathrm{Spec}\,(\bigoplus_{a,b} \mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b)[-a], d_1),$$

which is now in $dg_{\mathbb{Z}}\mathrm{Aff}(R)_*(\bar{S})$, since $\mathbb{O}(R)$ is a sum of weight 0 VHS, making $\mathrm{H}^{a-b}(D^{(b)}, \mathbb{O}(R) \otimes_{\mathbb{R}} \varepsilon^b)$ a weight $a-b+2b = a+b$ Hodge structure, and hence an $\bar{S}$-representation. ☐

Proposition 2.23. *If the discrete S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$ descends to R, then there are natural $(S^1)^\delta$-actions on $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ and $\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$, compatible with the opposedness isomorphism, and with $-1 \in S^1$ acting as $-1 \in \mathbb{G}_m$.*

Proof. This is a direct consequence of Proposition 2.12 and Proposition 2.20, since the Rees module construction transfers the discrete S^1-action. $\square$

2.4. **Singular and simplicial varieties.**

Proposition 2.24. *If Y is any separated complex scheme of finite type, there exists a simplicial smooth proper complex variety $X_\bullet$, a simplicial divisor $D_\bullet \subset X_\bullet$ with normal crossings, and a map $(X_\bullet - D_\bullet) \to Y$ such that $|X_\bullet - D_\bullet| \to Y$ is a weak equivalence, where $|Z_\bullet|$ is the geometric realisation of the simplicial space $Z_\bullet(\mathbb{C})$.*

Proof. The results in [Del2, §8.2] and [SD, Propositions 5.1.7 and 5.3.4], adapted as in [Pri4, Corollary 9.3], give the equivalence required. $\square$

Definition 2.25. Given a simplicial diagram $X_\bullet$ of smooth proper varieties and a point $x \in X_0$, define the fundamental group $\varpi_1(|X_\bullet|, x)^{\mathrm{norm}}$ to be the quotient of $\varpi_1(|X_\bullet|, x)$ by the normal subgroup generated by the image of $\mathrm{R_u}\varpi_1(X_0, x)$. We call its representations normally semisimple local systems on $|X_\bullet|$ — these correspond to local systems $\mathbb{V}$ (on the connected component of $|X|$ containing x) for which $a_0^{-1}\mathbb{V}$ is semisimple, for $a_0 : X_0 \to |X_\bullet|$.

Then define $\varpi_1(|X_\bullet|, x)^{\mathrm{norm,red}}$ to be the reductive quotient of $\varpi_1(|X_\bullet|, x)^{\mathrm{norm}}$. Its representations are semisimple and normally semisimple local systems on the connected component of $|X|$ containing x.

Definition 2.26. If $X_\bullet \to X$ is any resolution as in Proposition 2.24, with $x_0 \in X_0$ mapping to $x \in X$, we denote the corresponding pro-algebraic group by $\varpi_1(X, x)^{\mathrm{norm}} := \varpi_1(|X_\bullet|, x_0)^{\mathrm{norm}}$, noting that this is independent of the choices $X_\bullet$ and x_0, by [Pri4, Lemma 9.5].

Proposition 2.27. *If X is a proper complex variety with a smooth proper resolution $a : X_\bullet \to X$, then normally semisimple local systems on $X_\bullet$ correspond to local systems on X which become semisimple on pulling back to the normalisation $\pi : X^{\mathrm{norm}} \to X$ of X.*

Proof. This is [Pri4, Proposition 9.7]. $\square$

Proposition 2.28. *If $X_\bullet$ is a simplicial diagram of compact Kähler manifolds, then there is a discrete action of the circle group S^1 on $\varpi_1(|X_\bullet|, x)^{\mathrm{norm}}$, such that the composition $S^1 \times \pi_1(X_\bullet, x) \to \varpi_1(|X_\bullet|, x)^{\mathrm{norm}}(\mathbb{R})$ is continuous. We denote this last map by $\sqrt{h} : \pi_1(|X_\bullet|, x) \to \varpi_1(|X_\bullet|, x)^{\mathrm{norm}}((S^1)^{\mathrm{cts}})$.*

This also holds if we replace $X_\bullet$ with any proper complex variety X.

Proof. This is [Pri4, Proposition 9.8]. $\square$

Definition 2.29. Recall that the Thom–Sullivan (or Thom–Whitney) functor Th from cosimplicial DG algebras to DG algebras is defined as follows. Let $\Omega(|\Delta^n|)$ be the DG algebra of rational polynomial forms on the n-simplex, so

$$\Omega(|\Delta^n|) = \mathbb{Q}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(1 - \sum_i t_i),$$

for t_i of degree 0. The usual face and degeneracy maps for simplices yield $\partial_i : \Omega(|\Delta^n|) \to \Omega(|\Delta^{n-1}|)$ and $\sigma_i : \Omega(|\Delta^n|) \to \Omega(|\Delta^{n-1}|)$, giving a simplicial DGA.

Given a cosimplicial DG algebra $A^{\bullet\bullet}$ (with the first index denoting cosimplicial structure and the second, DG) , we then set

$$\mathrm{Th}\,(A) := \{a \in \prod_n A^{n\bullet} \otimes \Omega(|\Delta^n|) \,:\, \partial^i_A a_n = \partial_i a_{n+1},\, \sigma^j_A a_n = \sigma_j a_{n-1}\, \forall i, j\}.$$

Now, let $X_\bullet$ be a simplicial smooth proper complex variety, and $D_\bullet \subset X_\bullet$ a simplicial divisor with normal crossings. Set $Y_\bullet = X_\bullet - D_\bullet$, assume that $|Y_\bullet|$ is connected, and pick a point $y \in |Y_\bullet|$. Let $j : |Y_\bullet| \to |X_\bullet|$ be the natural inclusion map.

Using Proposition 2.24, the following gives mixed twistor or mixed Hodge structures on relative Malcev homotopy types of arbitrary complex varieties.

Theorem 2.30. *If R is any quotient of $\varpi_1(|X_\bullet|, jy)^{\mathrm{norm,red}}$ (resp. any quotient to which the $(S^1)^\delta$-action of Proposition 2.28 descends and acts algebraically), then there is an algebraic mixed twistor structure (resp. mixed Hodge structure) $(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) on the relative Malcev homotopy type $(|Y_\bullet|, y)^{R,\mathrm{Mal}}$.*

There is also a canonical $\mathbb{G}_m$-equivariant (resp. S-equivariant) splitting

$$\mathbb{A}^1 \times (\underline{\mathrm{gr}}(|Y_\bullet|^{R,\mathrm{Mal}}, 0)_{\mathrm{MTS}}) \times \mathrm{SL}_2 \simeq (|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{C^*,\mathrm{row}_1} \mathrm{SL}_2$$

(resp.

$$\mathbb{A}^1 \times (\underline{\mathrm{gr}}(|Y_\bullet|^{R,\mathrm{Mal}}, 0)_{\mathrm{MHS}}) \times \mathrm{SL}_2 \simeq (|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \times^{\mathbf{R}}_{C^*,\mathrm{row}_1} \mathrm{SL}_2)$$

on pulling back along $\mathrm{row}_1 : \mathrm{SL}_2 \to C^*$, whose pullback over $0 \in \mathbb{A}^1$ is given by the opposedness isomorphism.

Proof. We adapt the proof of [Pri4, Theorem 9.12]. Define the cosimplicial DGA $\tilde{A}(X_\bullet, \mathbb{O}(R))[\![D_\bullet]\!]$ on C by $n \mapsto \tilde{A}^\bullet(X_n, \mathbb{O}(R))[\![D_n]\!]$, observing that functoriality ensures that the cosimplicial and DGA structures are compatible. This has an augmentation $(jy)^* : \tilde{A}(X_\bullet, \mathbb{O}(R))[\![D_\bullet]\!] \to O(R) \otimes O(C)$ given in level n by $((\sigma_0)^n x)^*$, and inherits a filtration J from the DGAs $\tilde{A}^\bullet(X_n, \mathbb{O}(R))[\![D_n]\!]$.

We then define the mixed Hodge structure to be the object of $dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1\times C^*}(\mathrm{Mat}_1 \times R \rtimes S)$ given by

$$|Y_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MHS}} := (\mathrm{Spec}\,\mathrm{Th}\,\xi(\tilde{A}(X_\bullet, \mathbb{O}(R))[\![D_\bullet]\!], \mathrm{Dec}\,\mathrm{Th}\,(J))) \times_C C^*.$$

$|Y_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ is defined similarly, replacing S with $\mathbb{G}_m$. The graded object is given by

$$\underline{\mathrm{gr}}|X_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MHS}} = \mathrm{Spec}\,(\mathrm{Th}\,\mathrm{H}^*(X_\bullet, \mathbb{O}(R))) \in dg_Z\mathrm{Aff}(R \rtimes \bar{S}),$$

with $\underline{\mathrm{gr}}|Y_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ given by replacing $\bar{S}$ with Mat_1.

For any DGA B, we may regard B as a cosimplicial DGA (with constant cosimplicial structure), and then $\mathrm{Th}\,(B) = B$. In particular, $\mathrm{Th}\,(O(R)) = O(R)$, so we have a basepoint $\mathrm{Spec}\,\mathrm{Th}\,((jy)^*) : \mathbb{A}^1 \times R \times C^* \to |Y_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MHS}}$, giving

$$(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \in dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1\times C^*}(R)_*(\mathrm{Mat}_1 \times S),$$

and similarly for $|Y_\bullet|^{R,\mathrm{Mal}}_{\mathrm{MTS}}$.

The proofs of Theorems 2.22 and 2.21 now carry over. □

3. Algebraic MHS/MTS for quasi-projective varieties II — non-trivial monodromy

In this section, we assume that X is a smooth projective complex variety, with $Y = X - D$ (for D still a divisor locally of normal crossings). The hypothesis in Theorems 2.21 and 2.22 that R be a quotient of $\varpi_1(X, jy)$ is unnecessarily strong, and corresponds to allowing only those semisimple local systems on Y with trivial monodromy around the divisor. By [Moc1], every semisimple local system on Y carries an essentially unique tame imaginary pluriharmonic metric, so it is conceivable that Theorem 2.21 could hold for any reductive quotient R of $\varpi_1(Y, y)$.

However, Simpson's discrete S^1-action on $\varpi_1(X, jy)^{\mathrm{red}}$ does not extend to the whole of $\varpi_1(Y, y)^{\mathrm{red}}$, but only to a quotient ${}^{\nu}\varpi_1(Y, y)^{\mathrm{red}}$. This is because given a tame pure imaginary Higgs form θ and $\lambda \in S^1$, the Higgs form $\lambda\theta$ is only pure imaginary if either $\lambda = \pm 1$ or θ is nilpotent. The group ${}^{\nu}\varpi_1(Y, y)^{\mathrm{red}}$ is characterised by the property that its

representations are semisimple local systems whose associated Higgs form has nilpotent residues. This is equivalent to saying that ${}^{\nu}\varpi_1(Y,y)^{\mathrm{red}}$-representations are semisimple local systems on Y for which the monodromy around any component of D has unitary eigenvalues. Thus the greatest generality in which Proposition 2.23 could possibly hold is for any S^1-equivariant quotient R of ${}^{\nu}\varpi_1(Y,y)^{\mathrm{red}}$.

Denote the maximal quotient of ${}^{\nu}\varpi_1(Y,y)^{\mathrm{red}}$ on which the S^1-action is algebraic by ${}^{\mathrm{VHS}}\varpi_1(Y,y)$. Arguing as in [Pri4, Proposition 5.12], representations of ${}^{\mathrm{VHS}}\varpi_1(Y,y)$ correspond to real local systems underlying variations of Hodge structure on Y, and representations of ${}^{\mathrm{VHS}}\varpi_1(Y,y) \rtimes S^1$ correspond to weight 0 real VHS. The greatest generality in which Theorem 2.22 could hold is for any S^1-equivariant quotient R of ${}^{\mathrm{VHS}}\varpi_1(Y,y)^{\mathrm{red}}$.

Definition 3.1. Given a semisimple real local system $\mathbb{V}$ on Y, use Mochizuki's tame imaginary pluriharmonic metric to decompose the associated connection $D : \mathscr{A}^0_X(\mathbb{V}) \to \mathscr{A}^1_X(\mathbb{V})$ as $D = d^+ + \vartheta$ into antisymmetric and symmetric parts, and let $D^c := i \diamond d^+ - i \diamond \vartheta$. Also write $D' = \partial + \bar{\theta}$ and $D'' = \bar{\partial} + \theta$. Note that these definitions are independent of the choice of pluriharmonic metric, since the metric is unique up to global automorphisms $\Gamma(X, \mathrm{Aut}(\mathbb{V}))$.

3.1. **Constructing mixed Hodge structures.** We now outline a strategy for adapting Theorem 2.22 to more general R.

Proposition 3.2. *Let R be a quotient of ${}^{\mathrm{VHS}}\varpi_1(Y,y)$ to which the S^1-action descends, and assume we have the following data.*

- *For each weight 0 real VHS $\mathbb{V}$ on Y corresponding to an $R \rtimes S^1$-representation, an S-equivariant $\mathbb{R}$-linear graded subsheaf*

$$\mathscr{T}^*(\mathbb{V}) \subset j_*\mathscr{A}^*_Y(\mathbb{V}) \otimes \mathbb{C},$$

  on X, closed under the operations D and D^c. This must be functorial in $\mathbb{V}$, with
  - *$\mathscr{T}^*(\mathbb{V} \oplus \mathbb{V}') = \mathscr{T}^*(\mathbb{V}) \oplus \mathscr{T}^*(\mathbb{V}')$,*
  - *the image of $\mathscr{T}^*(\mathbb{V}) \otimes \mathscr{T}^*(\mathbb{V}') \xrightarrow{\wedge} j_*\mathscr{A}^*_Y(\mathbb{V} \otimes \mathbb{V}') \otimes \mathbb{C}$ contained in $\mathscr{T}^*(\mathbb{V} \otimes \mathbb{V}')$, and*
  - *$1 \in \mathscr{T}^*(\mathbb{R})$.*
- *An increasing non-negative S-equivariant filtration J of $\mathscr{T}^*(\mathbb{V})$ with $J_r\mathscr{T}^n(\mathbb{V}) = \mathscr{T}^n(\mathbb{V})$ for all $n \le r$, compatible with the tensor structures, and closed under the operations D and D^c.*

Set $F^p\mathscr{T}^\bullet(\mathbb{V}) := \mathscr{T}^\bullet(\mathbb{V}) \cap F^p\mathscr{A}^\bullet(Y,\mathbb{V})_{\mathbb{C}}$, where the Hodge filtration F is defined in the usual way in terms of the S-action, and assume that

1. *The map $\mathscr{T}^\bullet(\mathbb{V}) \to j_*\mathscr{A}^\bullet_Y(\mathbb{V})_{\mathbb{C}}$ is a quasi-isomorphism of sheaves on X for all $\mathbb{V}$.*
2. *For all $i \ne r$, the sheaf $\mathscr{H}^i(\mathrm{gr}^J_r\mathscr{T}^\bullet(\mathbb{V}))$ on X is 0.*
3. *For all a, b and p, the map*

$$\mathbb{H}^{a+b}(X, F^p\mathrm{gr}^J_b\mathscr{T}^\bullet(\mathbb{V})) \to \mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{V})_{\mathbb{C}}$$

   is injective, giving a Hodge filtration $F^p\mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{V})_{\mathbb{C}}$ which defines a pure ind-Hodge structure of weight $a + 2b$ on $\mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{V})$.

Then there is a non-negatively weighted mixed Hodge structure $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$, with

$$\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \simeq \mathrm{Spec}\,(\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{O}(R))[-a-b], d_2),$$

where $\mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{O}(R))$ naturally becomes a pure Hodge structure of weight $a + 2b$, and $d_2\colon \mathrm{H}^a(X, \mathbf{R}^b j_*\mathbb{O}(R)) \to \mathrm{H}^{a+2}(X, \mathbf{R}^{b-1} j_*\mathbb{O}(R))$ is the differential from the E_2 sheet of the Leray spectral sequence for j.

Proof. We proceed along similar lines to [Mor]. To construct the Hodge filtration, we first define $\tilde{\mathscr{T}}^\bullet(\mathbb{V}) \subset j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{V})_{\mathbb{C}}$ to be the subcomplex on the graded sheaf $\mathscr{T}^*(\mathbb{V}) \otimes O(C)$, then let $\mathscr{E}_{\mathbb{F}}(\mathbb{O}(R))$ be the homotopy fibre product

$$(\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R)) \otimes_{O(C)\otimes\mathbb{C}} O(\widetilde{C^*})) \times^h_{(j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))\otimes_{O(C)}O(S)\otimes\mathbb{C})} (j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)) \otimes_{O(C)} O(S))$$

in the category of $R \rtimes S$-equivariant DGAs on $X \times C^*_{\mathrm{Zar}}$, quasi-coherent over C^*. Here, we are extending $\mathscr{T}^\bullet$ to ind-VHS by setting $\mathscr{T}^\bullet(\varinjlim_\alpha \mathbb{V}_\alpha) := \varinjlim_\alpha \mathscr{T}^\bullet(\mathbb{V}_\alpha)$, and similarly for $\tilde{\mathscr{T}}^\bullet$.

Explicitly, a homotopy fibre product $C \times^h_D F$ is defined by replacing $C \to D$ with a quasi-isomorphic surjection $C' \twoheadrightarrow D$, then setting $C \times^h_D F := C' \times_D F$. Equivalently, we could replace $F \to D$ with a surjection. That such surjections exist and give well-defined homotopy fibre products up to quasi-isomorphism follows from the observation in [Pri4, Proposition 3.45] that the homotopy category of quasi-coherent DGAs on a quasi-affine scheme can be realised as the homotopy category of a right proper model category.

Observe that for co-ordinates u, v on C as in Definition 1.3,

$$\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R)) \otimes_{O(C)\otimes\mathbb{C}} O(\widetilde{C^*}) \cong (\bigoplus_{p\in\mathbb{Z}} F^p\mathscr{T}^\bullet(\mathbb{O}(R))(u+iv)^{-p})[(u-iv),(u-iv)^{-1}],$$

while $(j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)) \otimes_{O(C)} O(S)) \cong j_*\mathscr{A}^\bullet_Y(\mathbb{O}(R)) \otimes O(S)$ (with the same reasoning as [Pri4, Lemma 2.4]).

Note that $\widetilde{C^*} \times_C \widetilde{C^*} \cong \widetilde{C^*} \sqcup S_{\mathbb{C}}$, so $\mathscr{E}_{\mathbb{F}}(\mathbb{O}(R))|_{\widetilde{C^*}}$ is

$$\begin{aligned}
&[\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}} \oplus \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{S_{\mathbb{C}}}] \times^h_{[j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}} \oplus j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}]} [j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}\\
\simeq&[\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}} \oplus j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}] \times^h_{[j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}} \oplus j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}]} [j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}\\
\simeq&\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}}.
\end{aligned}$$

Similarly, $\mathscr{E}_{\mathbb{F}}(\mathbb{O}(R))|_S \simeq j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)) \otimes_{O(C)} O(S)$.

If we let $\mathrm{C}^\bullet(X,-)$ denote either the cosimplicial Čech or Godement resolution on X, then the Thom–Sullivan functor Th of Definition 2.29 gives us a composition $\mathrm{Th}\circ\mathrm{C}^\bullet(X,-)$ from sheaves of DG algebras on X to DG algebras. We denote this by $\mathbf{R}\Gamma(X,-)$, since it gives a canonical choice for derived global sections. We then define the Hodge filtration by

$$\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{F}} := \mathbf{R}\Gamma(X, \mathscr{E}_{\mathbb{F}}(\mathbb{O}(R)))$$

as an object of $\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{C^*}(R)_*(S))$. Note that condition (1) above ensures that the pullback of $(Y,y)^{R,\mathrm{Mal}}_{\mathbb{F}}$ over $1 \in C^*$ is quasi-isomorphic to $\mathrm{Spec}\,\mathbf{R}\Gamma(X, j_*\mathscr{A}^\bullet_Y(\mathbb{O}(R))$. Since the map

$$A^\bullet(Y,\mathbb{O}(R)) \to \mathbf{R}\Gamma(X, j_*\mathscr{A}^\bullet_Y(\mathbb{O}(R))$$

is a quasi-isomorphism, this means that $(Y,y)^{R,\mathrm{Mal}}_{\mathbb{F}}$ indeed defines an algebraic Hodge filtration on $(Y,y)^{R,\mathrm{Mal}}$.

To define the mixed Hodge structure, we first note that condition (2) above implies that

$$(\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R)) \otimes_{O(C)} O(S), \tau) \to (\tilde{\mathscr{T}}^\bullet(\mathbb{O}(R)) \otimes_{O(C)} O(S), J)$$

is a filtered quasi-isomorphism of complexes, where τ denotes the good truncation filtration. We then define $\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ to be the homotopy limit of the diagram

$$\begin{tikzcd}
\xi(\mathbf{R}\Gamma(X, \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}}), \mathrm{Dec}\,\mathbf{R}\Gamma(J)) \arrow[r] & \xi(\mathbf{R}\Gamma(X, \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{S_{\mathbb{C}}}), \mathrm{Dec}\,\mathbf{R}\Gamma(J)) \\
\xi(\mathbf{R}\Gamma(X, \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{S_{\mathbb{C}}}), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau)) \arrow[r] \arrow[ur] & \xi(\mathbf{R}\Gamma(X, j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_{S_{\mathbb{C}}}), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau)) \\
\xi(\mathbf{R}\Gamma(X, j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))|_S), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau)) \arrow[ur] &
\end{tikzcd}$$

which can be expressed as an iterated homotopy fibre product of the form $E_1 \times^h_{E_2} E_3 \times^h_{E_4} E_5$. Here, ξ denotes the Rees algebra construction as in Lemma 1.1. The basepoint $jy \in X$ gives an augmentation of this DG algebra, so we have defined an object of $\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times S))$.

Conditions (2) and (1) above ensure that the second and third maps in the diagram above are both quasi-isomorphisms, with the second map becoming an isomorphism on pulling back along $1 \in \mathbb{A}^1$ (corresponding to forgetting the filtrations). The latter observation means that we do indeed have

$$(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \times^{\mathbf{R}}_{\mathbb{A}^1,1} \mathrm{Spec}\,\mathbb{R} \simeq (Y,y)^{R,\mathrm{Mal}}_{\mathbb{F}}.$$

Setting $\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ as in the statement above, it only remains to establish opposedness.

Now, the pullback of $\xi(M,W)$ along $0 \in \mathbb{A}^1$ is just $\mathrm{gr}^W M$. Moreover, [Del1, Proposition 1.3.4] shows that for any filtered complex (M,J), the map

$$\mathrm{gr}^{\mathrm{Dec}\,J} M \to (\bigoplus_{a,b} \mathrm{H}^a(\mathrm{gr}^J_b M)[-a], d_1^J)$$

is a quasi-isomorphism, where d_1^J is the differential in the E_1 sheet of the spectral sequence associated to J. Thus the structure sheaf $\mathscr{G}$ of $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R}$ is the homotopy limit of the diagram

$$\begin{array}{l}
(\bigoplus_{a,b} \mathbb{H}^a(X, \mathrm{gr}^J_b \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}})[-a], d_1^J) \longrightarrow (\bigoplus_{a,b} \mathbb{H}^a(X, \mathrm{gr}^J_b \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{S_{\mathbb{C}}})[-a], d_1^J) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nearrow \\
(\bigoplus_{a,b} \mathrm{H}^a(X, \mathscr{H}^b \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{S_{\mathbb{C}}})[-a], d_2) \longrightarrow (\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S_{\mathbb{C}}})[-a], d_2) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nearrow \\
(\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S})[-a], d_2)
\end{array}$$

where d_2 denotes the differential on the E_2 sheet of the spectral sequence associated to a bigraded complex.

The second and third maps in the diagram above are isomorphisms, so we can write $\mathscr{G}$ as the homotopy fibre product of

$$\begin{array}{l}
(\bigoplus_{a,b} \mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \tilde{\mathscr{T}}^\bullet(\mathbb{O}(R))|_{\widetilde{C^*}})[-a-b], d_1^J) \longrightarrow (\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S_{\mathbb{C}}})[-a-b], d_2) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nearrow \\
(\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))|_{S})[-a-b], d_2)
\end{array}$$

By condition (3) above, $\mathbb{H}^a(X, \mathbf{R}^b j_*(\mathbb{O}(R))$ has the structure of an S-representation of weight $a+2b$ — denote this by E^{ab}, and set $E := (\bigoplus_{a,b} E^{ab}, d_2)$. Then we can apply Lemma 1.8 to rewrite $\mathscr{G}$ as

$$(\bigoplus_{p\in\mathbb{Z}} F^p(E\otimes\mathbb{C})(u+iv)^{-p})[(u-iv),(u-iv)^{-1}] \times^h_{E\otimes O(S_{\mathbb{C}})} E\otimes O(S).$$

Since $\bigoplus_{p\in\mathbb{Z}} F^p(E\otimes\mathbb{C})(u+iv)^{-p})[(u-iv),(u-iv)^{-1}] \cong E\otimes O(\widetilde{C^*})$, this is just

$$E\otimes(O(\widetilde{C^*}) \times^h_{O(S_{\mathbb{C}})} O(S)) \simeq E\otimes\mathscr{O}(C^*),$$

as required. □

3.2. **Constructing mixed twistor structures.** Proposition 3.2 does not easily adapt to mixed twistor structures, since an S-equivariant morphism $M \to N$ of quasi-coherent sheaves on S is an isomorphism if and only if the fibres $M_1 \to N_1$ are isomorphisms of vector spaces, but the same is not true of a $\mathbb{G}_m$-equivariant morphism of quasi-coherent sheaves on S. Our solution is to introduce holomorphic properties, the key idea being that for t the co-ordinate on S^1, the connection $t \circledast D\colon \mathscr{A}^0_Y(\mathbb{V})\otimes O(S^1) \to \mathscr{A}^1_Y(\mathbb{V})\otimes O(S^1)$ does not define a local system of $O(S^1)$-modules, essentially because iterated integration takes

us outside $O(S^1)$. However, as observed in [Sim1, end of §3], $t \circledast D$ defines a holomorphic family of local systems on X, parametrised by $S^1(\mathbb{C}) = \mathbb{C}^\times$.

Definition 3.3. Given a smooth complex affine variety Z, define $O(Z)^{\mathrm{hol}}$ to be the ring of holomorphic functions $f\colon Z(\mathbb{C}) \to \mathbb{C}$. Given a smooth real affine variety Z, define $O(Z)^{\mathrm{hol}}$ to be the ring of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant holomorphic functions $f\colon Z(\mathbb{C}) \to \mathbb{C}$.

In particular, $O(S^1)^{\mathrm{hol}}$ is the ring of functions $f : \mathbb{C}^\times \to \mathbb{C}$ for which $\overline{f(z)} = f(\bar{z}^{-1})$, or equivalently convergent Laurent series $\sum_{n\in\mathbb{Z}} a_n t^n$ for which $\bar{a}_n = a_{-n}$.

Definition 3.4. Given a smooth complex variety Z, define $\mathscr{A}^0_Y\mathscr{O}^{\mathrm{hol}}_Z$ to be the sheaf on $Y \times Z(\mathbb{C})$ consisting of smooth complex functions which are holomorphic along Z. Write $\mathscr{A}^\bullet_Y\mathscr{O}^{\mathrm{hol}}_Z := \mathscr{A}^\bullet_Y \otimes_{\mathscr{A}^0_Y} \mathscr{A}^0_Y\mathscr{O}^{\mathrm{hol}}_Z$, and, given a local system $\mathbb{V}$ on Y, set $\mathscr{A}^\bullet_Y\mathscr{O}^{\mathrm{hol}}_Z(\mathbb{V}) := \mathscr{A}^\bullet_Y(\mathbb{V}) \otimes_{\mathscr{A}^0_Y} \mathscr{A}^0_Y\mathscr{O}^{\mathrm{hol}}_Z$.

Given a smooth real variety Z, define $\mathscr{A}^0_Y\mathscr{O}^{\mathrm{hol}}_Z$ to be the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf $\mathscr{A}^0_Y\mathscr{O}^{\mathrm{hol}}_{Z_\mathbb{C}}$ on $Y \times Z(\mathbb{C})$, where the the non-trivial element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts by $\sigma(f)(y,z) = \overline{f(y, \sigma z)}$.

Definition 3.5. Define $P := C^*/\mathbb{G}_m$ and $\tilde{P} := \widetilde{C^*}/\mathbb{G}_m$. As in Definition 1.19, we have $S^1 = S/\mathbb{G}_m$, and hence a canonical inclusion $S^1 \hookrightarrow P$ (given by cutting out the divisor $\{(u:v) : u^2 + v^2 = 0\}$). For co-ordinates u, v on C as in Definition 1.3, fix co-ordinates $t = \frac{u+iv}{u-iv}$ on $\tilde{P}$, and $a = \frac{u^2-v^2}{u^2+v^2}$, $b = \frac{2uv}{u^2+v^2}$ on S^1 (so $a^2 + b^2 = 1$).

Thus $P \cong \mathbb{P}^1_\mathbb{R}$ and $\tilde{P} \cong \mathbb{A}^1_\mathbb{C}$, the latter isomorphism using the co-ordinate t. The canonical map $\tilde{P} \to P$ is given by $t \mapsto (1+t : i-it)$, and the map $S^1_\mathbb{C} \to \tilde{P}$ by $(a,b) \mapsto a+ib$.

Also note that the étale pushout $C^* = \widetilde{C^*} \cup_{S_\mathbb{C}} S$ corresponds to an étale pushout

$$P = \tilde{P} \cup_{S^1_\mathbb{C}} S^1,$$

where $S^1_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}}$ is given by the subscheme $t \neq 0$ in $\mathbb{A}^1_\mathbb{C}$. Note that the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on $\mathbb{C}[t, t^{-1}]$ given by the real form S^1 is determined by the condition that the non-trivial element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ maps t to t^{-1}.

Definition 3.6. Define $\breve{\mathscr{A}}^\bullet_Y(\mathbb{V})$ to be the sheaf $\bigoplus_{n\geq 0} \mathscr{A}^n_Y(\mathbb{V})\mathscr{O}^{\mathrm{hol}}_P(n)$ of graded algebras on $Y \times P(\mathbb{C})$, equipped with the differential $uD + vD^c$, where $u, v \in \Gamma(P, \mathscr{O}_P(1))$ correspond to the weight 1 generators $u, v \in O(C)$.

Definition 3.7. Given a polarised scheme $(Z, \mathscr{O}_Z(1))$ (where Z need not be projective), and a sheaf $\mathscr{F}$ of $\mathscr{O}_Z$-modules, define $\underline{\Gamma}(\mathscr{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(Z, \mathscr{F}(n))$. This is regarded as a $\mathbb{G}_m$-representation, with $\Gamma(Z, \mathscr{F}(n))$ of weight n.

Lemma 3.8. *The $\mathbb{G}_m$-equivariant sheaf $\tilde{\mathscr{A}}^\bullet_Y(\mathbb{V})$ of $O(C)$-complexes on Y (from Definition 2.7) is given by*

$$\tilde{\mathscr{A}}^\bullet_Y(\mathbb{V}) \cong \underline{\Gamma}(P(\mathbb{C}), \breve{\mathscr{A}}^\bullet_Y(\mathbb{V})(n))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}.$$

Proof. We first consider $\Gamma(P(\mathbb{C}), \breve{\mathscr{A}}^0_Y(\mathbb{V}))$. This is the sheaf on Y which sends any open subset $U \subset Y$ to the ring of consisting of those smooth functions $f\colon U \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}$ which are holomorphic along $\mathbb{P}^1(\mathbb{C})$. Thus for any $y \in U$, $f(y, -)$ is a global holomorphic function on $\mathbb{P}^1(\mathbb{C})$, so is constant. Therefore $\Gamma(P(\mathbb{C}), \breve{\mathscr{A}}^0_Y(\mathbb{V})) = \mathscr{A}^0_Y \otimes \mathbb{C}$.

For general n, a similar argument using finite-dimensionality of $\Gamma(\mathbb{P}^1(\mathbb{C}), \mathscr{O}(n)^{\mathrm{hol}})$ shows that

$$\Gamma(P(\mathbb{C}), \breve{\mathscr{A}}^0_Y(\mathbb{V})(n)) \cong \mathscr{A}^0_Y \otimes \Gamma(P(\mathbb{C}), \mathscr{O}_P(n)^{\mathrm{hol}}).$$

Now by construction of P, we have $\underline{\Gamma}(P(\mathbb{C}), \mathscr{O}_P(n)^{\mathrm{hol}}) \cong O(C) \otimes \mathbb{C}$ with the grading corresponding the the $\mathbb{G}_m$-action. Thus

$$\underline{\Gamma}(P(\mathbb{C}), \breve{\mathscr{A}}^*_Y(\mathbb{V})(n))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \cong \tilde{\mathscr{A}}^*_Y(\mathbb{V}).$$

Since the differential in both cases is given by $uD+vD^c$, this establishes the isomorphism of complexes. $\square$

Definition 3.9. On the schemes S^1 and $\tilde{P}$, define the sheaf $\mathscr{O}(1)$ by pulling back $\mathscr{O}_P(1)$ from P. Thus the corresponding module $A(1)$ on $\operatorname{Spec} A$ is given by

$$A(1) = A(u,v)/(t(u-iv)-(u+iv)).$$

Hence $\mathscr{O}_{\tilde{P}}(1) = \mathscr{O}_{\tilde{P}}(u-iv)$ and $\mathscr{O}_{S^1}(1)\otimes\mathbb{C} = \mathscr{O}_{S^1}\otimes\mathbb{C}(u-iv)$ are trivial line bundles, but $\mathscr{O}_{S^1}(1) = \mathscr{O}_{S^1}(u,v)/(au+bv-u, bu-av-v)$.

Proposition 3.10. *Let R be a quotient of $\varpi_1(Y,y)^{\mathrm{red}}$, and assume that we have the following data.*

- *For each finite rank local real system $\mathbb{V}$ on Y corresponding to an R-representation, a flat $\mathscr{A}^0_X$-submodule graded subsheaf*

$$\mathscr{T}^*(\mathbb{V}) \subset j_*\mathscr{A}^*_Y(\mathbb{V})\otimes\mathbb{C},$$

  closed under the operations D and D^c. This must be functorial in $\mathbb{V}$, with
  - $\mathscr{T}^*(\mathbb{V}\oplus\mathbb{V}') = \mathscr{T}^*(\mathbb{V})\oplus\mathscr{T}^*(\mathbb{V}')$,
  - *the image of $\mathscr{T}^*(\mathbb{V})\otimes\mathscr{T}^*(\mathbb{V}') \xrightarrow{\wedge} j_*\mathscr{A}^*_Y(\mathbb{V}\otimes\mathbb{V}')\otimes\mathbb{C}$ contained in $\mathscr{T}^*(\mathbb{V}\otimes\mathbb{V}')$, and*
  - $1\in\mathscr{T}^*(\mathbb{R})$.
- *An increasing non-negative filtration J of $\mathscr{T}^*(\mathbb{V})$ with $J_r\mathscr{T}^n(\mathbb{V}) = \mathscr{T}^n(\mathbb{V})$ for all $n\le r$, compatible with the tensor structure, and closed under the operations D and D^c.*

Set $\breve{\mathscr{T}}^\bullet(\mathbb{V}) \subset j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V})$ to be the complex on $X\times P(\mathbb{C})$ whose underlying sheaf is $\bigoplus_{n\ge0}\mathscr{T}^n(\mathbb{V})\otimes_{\mathscr{A}^0_X}\mathscr{A}^0_X\mathscr{O}_P^{\mathrm{hol}}(n)$, and assume that

1. *For $S^1(\mathbb{C})\subset P(\mathbb{C})$, the map $\breve{\mathscr{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})} \to j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V})|_{S^1(\mathbb{C})}$ is a quasi-isomorphism of sheaves of $\mathscr{O}^{\mathrm{hol}}_{S^1}$-modules on $X\times S^1(\mathbb{C})$ for all $\mathbb{V}$.*
2. *For all $i\ne r$, the sheaf $\mathscr{H}^i(\mathrm{gr}^J_r\breve{\mathscr{T}}^\bullet(\mathbb{V})|_S)$ of $\mathscr{O}^{\mathrm{hol}}_{S^1}$-modules on $X\times S^1(\mathbb{C})$ is 0.*
3. *For all $a,b\ge0$, the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf*

$$\ker(\mathbb{H}^a(X,\mathrm{gr}^J_b\breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}\oplus\sigma^*\overline{\mathbb{H}^a(X,\mathrm{gr}^J_b\breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \to \mathrm{H}^a(X,\mathscr{H}^b(j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V})))|_{S^1(\mathbb{C})})$$

   is a finite locally free $\mathscr{O}^{\mathrm{hol}}_P$-module of slope $a+2b$.

Then there is a non-negatively weighted mixed twistor structure $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$, with

$$\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \simeq \operatorname{Spec}(\bigoplus_{a,b}\mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{O}(R))[-a-b],d_2),$$

where $\mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{O}(R))$ is assigned the weight $a+2b$, and $d_2\colon \mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{O}(R)) \to \mathrm{H}^{a+2}(X,\mathbf{R}^{b-1}j_*\mathbb{O}(R))$ is the differential from the E_2 sheet of the Leray spectral sequence for j.

Proof. Define $\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}}$ to be the homotopy fibre product

$$\mathbf{R}\Gamma(X,\underline{\Gamma}(\tilde{P}(\mathbb{C}),\breve{\mathscr{T}}^\bullet(\mathbb{O}(R))))\times^h_{\mathbf{R}\Gamma(X,j_*\underline{\Gamma}(S^1(\mathbb{C}),\breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))))}\mathbf{R}\Gamma(X,j_*\underline{\Gamma}(S^1(\mathbb{C}),\breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$$

as an object of $\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{C^*}(R)_*(\mathbb{G}_m))$, and let $\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ be the homotopy limit of the diagram

$$\begin{array}{c}
\xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(\tilde{P}(\mathbb{C}), \breve{\mathscr{T}}^\bullet(\mathbb{O}(R)))), \mathrm{Dec}\,\mathbf{R}\Gamma(J)) \\
\downarrow \\
\xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(S^1(\mathbb{C}), \breve{\mathscr{T}}^\bullet(\mathbb{O}(R)))), \mathrm{Dec}\,\mathbf{R}\Gamma(J)) \\
\uparrow \\
\xi(\mathbf{R}\Gamma(X, \underline{\Gamma}(S^1(\mathbb{C}), \breve{\mathscr{T}}^\bullet(\mathbb{O}(R)))), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau)) \\
\downarrow \\
\xi(\mathbf{R}\Gamma(X, j_*\underline{\Gamma}(S^1(\mathbb{C}), \breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)))), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau)) \\
\uparrow \\
\xi(\mathbf{R}\Gamma(X, j_*\underline{\Gamma}(S^1(\mathbb{C}), \breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}), \mathrm{Dec}\,\mathbf{R}\Gamma(\tau))
\end{array}$$

as an object of $\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1\times C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m))$. Here, we are extending $\mathscr{T}^\bullet$ to ind-local systems by setting $\mathscr{T}^\bullet(\varinjlim_\alpha \mathbb{V}_\alpha) := \varinjlim_\alpha \mathscr{T}^\bullet(\mathbb{V}_\alpha)$, and similarly for $\breve{\mathscr{T}}^\bullet$.

Given a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf $\mathscr{F}$ of $\mathscr{O}_P^{\mathrm{hol}}$-modules on $X \times P(\mathbb{C})$, the group cohomology complex gives a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant cosimplicial sheaf $\mathrm{C}^\bullet(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathscr{F})$ on $X \times P(\mathbb{C})$ — this is a resolution of $\mathscr{F}$, with $\mathrm{C}^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathscr{F}) = \mathscr{F} \oplus \sigma^*\bar{\mathscr{F}}$. Applying the Thom–Whitney functor Th, this means that

$$\mathrm{Th}\,\mathrm{C}^\bullet(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V}))$$

is a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant $\mathscr{O}_P^{\mathrm{hol}}$-DGA on $X \times P(\mathbb{C})$, equipped with a surjection to $j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V}) \oplus \sigma^*\overline{j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{V})}$.

This allows us to consider the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf $\mathscr{B}^\bullet_{\mathbb{T}}$ of $\mathscr{O}_P^{\mathrm{hol}}$-DGAs on $P(\mathbb{C})$ given by the fibre product of

$$\begin{array}{l}
(\breve{\mathscr{T}}^\bullet(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^*\overline{\breve{\mathscr{T}}^\bullet(\mathbb{O}(R))}|_{\tilde{P}(\mathbb{C})}) \longrightarrow (j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)) \oplus \sigma^*\overline{j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R))})|_{S^1(\mathbb{C})} \\
\mathrm{Th}\,\mathrm{C}^\bullet(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), j_*\breve{\mathscr{A}}^\bullet_Y(\mathbb{O}(R)))|_{S^1(\mathbb{C})} \nearrow.
\end{array}$$

Note that since the second map is surjective, this fibre product is in fact a homotopy fibre product. In particular,

$$\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}} \simeq \mathbf{R}\Gamma(X, \underline{\Gamma}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})})|_{C^*}.$$

Now, $\underline{\Gamma}(P(\mathbb{C}), -)$ gives a functor from Zariski sheaves o $\mathscr{O}_P^{\mathrm{hol}}$-modules to $O(C)$-modules, and we consider the functor $\underline{\Gamma}(P(\mathbb{C}), -)|_{C^*}$ to quasi-coherent sheaves on C^*. There is a right derived functor $\mathbf{R}\underline{\Gamma}(P(\mathbb{C}), -)$; by [Ser], the map

$$\underline{\Gamma}(P(\mathbb{C}), \mathscr{F})|_{C^*} \to \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathscr{F})|_{C^*}$$

is a quasi-isomorphism for all coherent $\mathscr{O}_P^{\mathrm{hol}}$-modules $\mathscr{F}$. Given a morphism $f : Z \to P_{\mathbb{C}}$ of polarised varieties, with Z affine, and a quasi-coherent Zariski sheaf of $\mathscr{O}_Z^{\mathrm{hol}}$-modules on Z, note that

$$\mathbf{R}\underline{\Gamma}(P(\mathbb{C}), f_*\mathscr{F}) \simeq \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathbf{R}f_*\mathscr{F}) \simeq \mathbf{R}\underline{\Gamma}(Z(\mathbb{C}), \mathscr{F}) \simeq \underline{\Gamma}(Z(\mathbb{C}), \mathscr{F}).$$

There are convergent spectral sequences

$$\mathrm{H}^a(P(\mathbb{C}), \mathscr{H}^b(\mathscr{B}^\bullet_{\mathbb{T}})(n)) \implies \mathbb{H}^{a+b}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}}(n))$$

for all n, and Condition (3) above ensures that $\mathscr{H}^b(\mathscr{B}^\bullet_{\mathbb{T}})$ is a direct sum of coherent sheaves. Since $\mathrm{H}^i\mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{H}^i(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}}(n))$, this means that the map

$$\underline{\Gamma}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}})|_{C^*} \to \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}})|_{C^*}$$

is a quasi-isomorphism. Combining these observations shows that

$$\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}} \simeq \mathbf{R}\Gamma(X, \mathbf{R}\underline{\Gamma}(P(\mathbb{C}), \mathscr{B}^\bullet_{\mathbb{T}}))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}|_{C^*}.$$

In particular,

$$\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}} \otimes^{\mathbf{L}}_{\mathscr{O}_{C^*}} O(\mathbb{G}_m) \to \mathbf{R}\Gamma(X, \underline{\Gamma}(\operatorname{Spec}\mathbb{C}, \mathscr{B}^{\bullet}_{\mathbb{T}} \otimes_{\mathscr{O}_P^{\mathrm{hol}},(1:0)} \mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}))$$

is a quasi-isomorphism, and note that right-hand side is just

$$\mathbf{R}\Gamma(X, (\mathscr{B}^{\bullet}_{\mathbb{T}} \otimes_{\mathscr{O}_P^{\mathrm{hol}},(1:0)} \mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}) \otimes O(\mathbb{G}_m),$$

which is the homotopy fibre

$$\mathbf{R}\Gamma(X, [\mathscr{T}^{\bullet}(\mathbb{O}(R)) \times^h_{j_*\mathscr{A}^{\bullet}_Y(\mathbb{O}(R))\otimes\mathbb{C}} j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R))]),$$

and hence quasi-isomorphic to $\mathbf{R}\Gamma(X, j_*\mathscr{A}^{\bullet}_Y(\mathbb{O}(R)))$ by condition (1) above. This proves that

$$(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}} \times^{\mathbf{R}}_{C^*,1} \operatorname{Spec}\mathbb{R} \simeq (Y,y)^{R,\mathrm{Mal}},$$

so $(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}}$ is indeed a twistor filtration on $(Y,y)^{R,\mathrm{Mal}}$.

The proof that $\mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathbb{T}} \simeq \mathscr{O}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathbb{A}^1},1} \operatorname{Spec}\mathbb{R}$ follows along exactly the same lines as in Proposition 3.2, so it only remains to establish opposedness.

Arguing as in the proof of Proposition 3.2, we see that the structure sheaf $\mathscr{G}$ of $\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \times^{\mathbf{R}}_{\mathbb{A}^1,0} \operatorname{Spec}\mathbb{R}$ is the homotopy fibre product of the diagram

$$\begin{array}{c}
(\bigoplus_{a,b} \underline{\Gamma}(\tilde{P}(\mathbb{C}), \mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R))))[-a-b], d_1^J) \\
\downarrow \\
(\bigoplus_{a,b} \underline{\Gamma}(S^1(\mathbb{C}), \mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}_Y(\mathbb{O}(R)))))[-a-b], d_2) \\
\uparrow \\
(\bigoplus_{a,b} \underline{\Gamma}(S^1(\mathbb{C}), \mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}_Y(\mathbb{O}(R)))))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}[-a-b], d_2),
\end{array}$$

as a $(\mathrm{Mat}_1 \times R \times \mathbb{G}_m)$-equivariant sheaf of DGAs over C^*.

Set $\underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}}$ to be the sheaf on $P(\mathbb{C})$ given by the fibre product of the diagram

$$\begin{array}{c}
\mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^*\overline{\mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R))|_{\tilde{P}(\mathbb{C})})} \\
\downarrow \\
\mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R)))) \oplus \sigma^*\overline{\mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R))))}|_{S^1(\mathbb{C})} \\
\uparrow \\
\operatorname{Th}\mathrm{C}^{\bullet}(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R)))))|_{S^1(\mathbb{C})},
\end{array}$$

and observe that

$$\mathscr{G} \simeq (\bigoplus_{a,b} \underline{\Gamma}(P(\mathbb{C}), \underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}|_{C^*}, d_1^J).$$

Now, $\underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}}$ is just the homotopy fibre product of

$$\begin{array}{c}
\mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^*\overline{\mathbb{H}^{a+b}(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R))|_{\tilde{P}(\mathbb{C})})} \\
\downarrow \\
\mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R)))) \oplus \sigma^*\overline{\mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R))))}|_{S^1(\mathbb{C})} \\
\uparrow \\
\mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R))))|_{S^1(\mathbb{C})};
\end{array}$$

condition (1) ensures that the first map is injective, so $\underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}}$ is quasi-isomorphic to the kernel of

$$\mathbb{H}^a(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^*\overline{\mathbb{H}^a(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^{\bullet}(\mathbb{O}(R)))|_{\tilde{P}(\mathbb{C})}} \to \mathrm{H}^a(X, \mathscr{H}^b(j_*\breve{\mathscr{A}}^{\bullet}_Y(\mathbb{O}(R))))|_{S^1(\mathbb{C})}.$$

By condition (3), this is a holomorphic vector bundle on $P(\mathbb{C})$ of slope $a+2b$.

Now, we just observe that for any holomorphic vector bundle $\mathscr{F}$ of slope m, the map $\Gamma(P(\mathbb{C}), \mathscr{F}(-m)) \to 1^*\mathscr{F}$, given by taking the fibre at $1 \in P(\mathbb{R})$, is an isomorphism of complex vector spaces, and that the maps

$$\Gamma(P(\mathbb{C}), \mathscr{F}(-m)) \otimes \Gamma(P(\mathbb{C}), \mathscr{O}(n)) \to \Gamma(P(\mathbb{C}), \mathscr{F}(n-m))$$

are isomorphisms for $n \geq 0$. This gives an isomorphism

$$\underline{\Gamma}(P(\mathbb{C}), \mathscr{F})|_{C^*} \cong (1^*\mathscr{F}) \otimes \mathscr{O}_{C^*}$$

over C^*, which becomes $\mathbb{G}_m$-equivariant if we set $1^*\mathscr{F}$ to have weight m.

Therefore

$$\underline{\Gamma}(P(\mathbb{C}), \underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}})|^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}_{C^*} \cong \mathrm{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \otimes \mathscr{O}_{C^*},$$

making use of condition (1) to show that $\mathrm{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R)) \otimes \mathbb{C}$ is the fibre of $\underline{\mathrm{gr}}\mathscr{B}^{a,b}_{\mathrm{MHS}}$ at $1 \in P(\mathbb{R})$. This completes the proof of opposedness. □

Proposition 3.11. *Let R be a quotient of ${}^{\nu}\varpi_1(Y,y)^{\mathrm{red}}$ to which the discrete S^1-action descends, assume that the conditions of Proposition 3.10 hold, and assume in addition that for all $\lambda \in \mathbb{C}^\times$, the map $\lambda\diamond\colon\ j_*\tilde{\mathscr{A}}^\bullet_Y(\mathbb{V}) \to j_*\tilde{\mathscr{A}}^\bullet_Y(\frac{\lambda}{\bar\lambda}\circledast\mathbb{V})$ maps $\mathscr{T}(\mathbb{V})$ isomorphically to $\mathscr{T}(\frac{\lambda}{\bar\lambda}\circledast \mathbb{V})$. Then there are natural $(S^1)^\delta$-actions on $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ and $\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$, compatible with the opposedness isomorphism, and with the action of $-1 \in S^1$ coinciding with that of $-1 \in \mathbb{G}_m$.*

Proof. The proof of Proposition 2.23 carries over, substituting Proposition 3.10 for Theorem 2.21. □

3.3. **Unitary monodromy.** In this section, we will consider only semisimple local systems $\mathbb{V}$ on Y with unitary monodromy around the local components of D (i.e. semisimple monodromy with unitary eigenvalues),

Definition 3.12. For $\mathbb{V}$ as above, let $\mathscr{M}(\mathbb{V}) \subset j_*\mathscr{A}^0(\mathbb{V})\otimes\mathbb{C}$ consist of locally L_2-integrable functions for the Poincaré metric, holomorphic in the sense that they lie in $\ker\bar\partial$, where $D = \partial + \bar\partial + \theta + \bar\theta$.

Then set

$$\mathscr{A}^*_X(\mathbb{V})\langle D\rangle := \mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_X} \mathscr{A}^*_X(\mathbb{R})\langle D\rangle \subset j_*\mathscr{A}_Y(\mathbb{V}) \otimes \mathbb{C},$$

where $\mathscr{O}_X$ denotes the sheaf of holomorphic functions on X.

The crucial observation which we now make is that $\mathscr{A}^*_X(\mathbb{V})\langle D\rangle$ is closed under the operations D and D^c. Closure under $\bar\partial$ is automatic, and closure under ∂ follows because Mochizuki's metric is tame, so $\partial\colon\ \mathscr{M}(\mathbb{V}) \to \mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_X} \Omega^1_X\langle D\rangle$. Since $\mathbb{V}$ has unitary monodromy around the local components of D, the Higgs form θ is holomorphic, which ensures that $\mathscr{A}^*_X(\mathbb{V})\langle D\rangle$ is closed under both θ and $\bar\theta$. We can thus write $\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle$ for the complex given by $\mathscr{A}^*_X(\mathbb{V})\langle D\rangle$ with differential D.

Lemma 3.13. *For all $m \geq 0$, there is a morphism*

$$\mathrm{Res}_m\colon\ \mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \to \nu_{m*}\mathscr{A}^\bullet_{D^{(m)}}(\nu_m^{-1}j_*\mathbb{V} \otimes \varepsilon^m)\langle C^{(m)}\rangle[-m],$$

compatible with both D and D^c, for $D^{(m)}, C^{(m)}$ as in Definition 2.13 .

Proof. As in [Tim2, 1.4], Res_m is given in level q by the composition

$$\begin{aligned}
\mathscr{A}^q_X(\mathbb{V})\langle D\rangle &= \mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_X} \mathscr{A}^q_X\langle D\rangle \\
&\xrightarrow{\mathrm{id}\otimes\mathrm{Res}_m} \mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_X} \nu_{m*}\mathscr{A}^{q-m}_{D^{(m)}}(\varepsilon^m_{\mathbb{R}})\langle C^{(m)}\rangle \\
&= \nu_{m*}[\varepsilon^m \otimes_{\mathbb{Z}} \nu_m^*\mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_{D^{(m)}}} \mathscr{A}^{q-m}_{D^{(m)}}(\varepsilon^m)\langle C^{(m)}\rangle \\
&\to \nu_{m*}[\varepsilon^m \otimes_{\mathbb{Z}} \nu_m^*\mathscr{M}(\mathbb{V}) \otimes_{\mathscr{O}_{D^{(m)}}} \mathscr{A}^{q-m}_{D^{(m)}}(\varepsilon^m)\langle C^{(m)}\rangle,
\end{aligned}$$

where the final map is given by orthogonal projection. The proof of [Tim2, Lemma 1.5] then adapts to show that $\mathrm{Res}_m$is compatible with both D and D^c. □

Note that $(j_*\mathbb{V}\otimes\varepsilon^m)|_{D^m-D^{m+1}}$ inherits a pluriharmonic metric from $\mathbb{V}$, so is necessarily a semisimple local system on the quasi-projective variety $D^m - D^{m+1} = D^{(m)} - C^{(m)}$.

Definition 3.14. Define a filtration on $\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle$ by

$$J_r\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle := \ker(\mathrm{Res}_{r+1}),$$

for $r \geq 0$. This generalises [Tim2, Definition 1.6].

Definition 3.15. Define the graded sheaf $\mathscr{L}^*_{(2)}(\mathbb{V})$ on X to consist of $j_*\mathbb{V}$-valued L^2 distributional forms a for which ∂a and $\bar\partial a$ are also L^2. Write $L^*_{(2)}(X,\mathbb{V}) := \Gamma(X,\mathscr{L}^*_{(2)}(\mathbb{V}))$.

Since θ is holomorphic, note that the operators θ and $\bar\theta$ are bounded, so also act on $\mathscr{L}^*_{(2)}(\mathbb{V})\otimes\mathbb{C}$.

3.3.1. *Mixed Hodge structures.*

Theorem 3.16. *If R is a quotient of ${}^{\mathrm{VHS}}\varpi_1(Y,y)$ for which the representation $\pi_1(Y,y)\to R(\mathbb{R})$ has unitary monodromy around the local components of D, then there is a canonical non-positively weighted mixed Hodge structure $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ on $(Y,y)^{R,\mathrm{Mal}}$, in the sense of Definition 1.23. The associated split MHS is given by*

$$\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \simeq \mathrm{Spec}\,(\bigoplus_{a,b}\mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{O}(R))[-a-b],d_2),$$

with $\mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{O}(R))$ a pure ind-Hodge structure of weight $a+2b$.

Proof. We apply Proposition 3.2, taking $\mathscr{T}^*(\mathbb{V}) := \mathscr{A}^*_X(\mathbb{V})\langle D\rangle$, equipped with its filtration J. The first condition to check is compatibility with tensor operations. This follows because, although a product of arbitrary L^2 functions is not L^2, a product of meromorphic L^2 functions is so.

Next, we check that $\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \to j_*\mathscr{A}^\bullet_Y(\mathbb{V})_{\mathbb{C}}$ is a quasi-isomorphism, with $\mathrm{gr}^J_m\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \simeq \mathbf{R}^mj_*\mathbb{V}[-m]$. [Tim2, Proposition 1.7] (which deals with unitary local systems), adapts to show that Res_m gives a quasi-isomorphism

$$\mathrm{gr}^J_m\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle \to J_0\nu_{m*}\mathscr{A}^\bullet_{D^{(m)}}(\nu_m^{-1}j_*\mathbb{V}\otimes\varepsilon^m)\langle C^{(m)}\rangle[-m].$$

Since $\mathbf{R}^mj_*\mathbb{V} \cong \nu_{m*}(\nu_m^{-1}j_*\mathbb{V}\otimes\varepsilon^m)$, this means that it suffices to establish the quasi-isomorphism for $m=0$ (replacing X with $D^{(m)}$ for the higher cases). The proof of [Tim1, Theorem D.2(a)] adapts to this generality, showing that $j_*\mathbb{V}\to J_0\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle$ is a quasi-isomorphism.

It only remains to show that for all a,b, the groups $\mathbb{H}^{a+b}(X,F^p\mathrm{gr}^J_b\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle)$ define a Hodge filtration on $\mathrm{H}^a(X,\mathbf{R}^bj_*\mathbb{V})_{\mathbb{C}}$, giving a pure ind-Hodge structure of weight $a+2b$. This is essentially [Tim2, Proposition 6.4]: the quasi-isomorphism induced above by Res_m is in fact a filtered quasi-isomorphism, provided we set ε^m to be of type (m,m). By applying a twist, we can therefore reduce to the case $b=0$ (replacing X with $D^{(b)}$ for the higher cases), so we wish to show that the groups $\mathbb{H}^a(X,F^pJ_0\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle)$ define a Hodge filtration on $\mathrm{H}^a(X,j_*\mathbb{V})$ of weight a.

The proof of [Tim1, Proposition D.4] adapts to give this result, by identifying $\mathrm{H}^*(X,j_*\mathbb{V})$ with L^2 cohomology, which in turn is identified with the space of harmonic forms. We have a bicomplex $(\Gamma(X,\mathscr{L}^*_{(2)}(\mathbb{V})\otimes\mathbb{C}),D',D'')$ satisfying the principle of two types, with $F^pJ_0\mathscr{A}^\bullet_X(\mathbb{V})\langle D\rangle\to F^p\mathscr{L}^\bullet_{(2)}(\mathbb{V})\otimes\mathbb{C}$ and $j_*\mathbb{V}\to\mathscr{L}^\bullet_{(2)}(\mathbb{V})$ both being quasi-isomorphisms. □

3.3.2. *Mixed twistor structures.*

Definition 3.17. Given a smooth complex variety Z, let $\mathscr{L}^*_{(2)}(\mathbb{V})\mathscr{O}^{\mathrm{hol}}_Z$ be the sheaf on $X \times Z(\mathbb{C})$ consisting of holomorphic families of L^2 distributions on X, parametrised by $Z(\mathbb{C})$. Explicitly, given a local co-ordinate z on $Z(\mathbb{C})$, the space $\Gamma(U \times \{|z| < R\}, \mathscr{L}^n_{(2)}(\mathbb{V})\mathscr{O}^{\mathrm{hol}}_P)$ consists of power series

$$\sum_{m \geq 0} a_m z^n$$

with $a_m \in \Gamma(U, \mathscr{L}^*_{(2)}(\mathbb{V})) \otimes \mathbb{C}$, such that for all $K \subset U$ compact and all $r < R$, the sum

$$\sum_{m \geq 0} \|a_m\|_{2,K} r^m$$

converges, where $\| - \|_{2,K}$ denotes the L^2 norm on K.

Definition 3.18. Set $\breve{L}^n_{(2)}(X, \mathbb{V})$ to be the complex of $\mathscr{O}^{\mathrm{hol}}_P$-modules on $P(\mathbb{C})$ given by

$$\breve{L}^n_{(2)}(X, \mathbb{V}) := \Gamma(X, \mathscr{L}^n_{(2)}(\mathbb{V})\mathscr{O}^{\mathrm{hol}}_P(n)),$$

with differential $uD + vD^c$. Note that locally on $P(\mathbb{C})$, elements of $\breve{L}^n_{(2)}(X, \mathbb{V})$ can be characterised as convergent power series with coefficients in $L^n_{(2)}(X, \mathbb{V}) \otimes \mathbb{C}$.

Theorem 3.19. *If $\pi_1(Y, y) \to R(\mathbb{R})$ is Zariski-dense, with unitary monodromy around the local components of D, then there is a canonical non-positively weighted mixed twistor structure $(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ on $(Y, y)^{R,\mathrm{Mal}}$, in the sense of Definition 1.24. The associated split MTS is given by*

$$\underline{\mathrm{gr}}(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \simeq \mathrm{Spec}\,(\bigoplus_{a,b} \mathrm{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a-b], d_2),$$

with $\mathrm{H}^a(X, \mathbf{R}^b j_* \mathbb{O}(R))$ of weight $a + 2b$.

Proof. We verify the conditions of Proposition 3.10, setting

$$\mathscr{T}^*(\mathbb{V}) \subset j_* \mathscr{A}_Y(\mathbb{V}) \otimes \mathbb{C}$$

to be $\mathscr{T}^*(\mathbb{V}) =: \mathscr{A}^*_X(\mathbb{V})\langle D \rangle$, with its filtration J defined above. This gives the complex $\breve{\mathscr{T}}^\bullet(\mathbb{V}) \subset j_* \breve{\mathscr{A}}^\bullet_Y(\mathbb{V})$ on $X \times P(\mathbb{C})$ whose underlying sheaf is $\bigoplus_{n \geq 0} \mathscr{T}^n(\mathbb{V}) \otimes_{\mathscr{A}^0_X} \mathscr{A}^0_X \mathscr{O}^{\mathrm{hol}}_P(n)$, with differential $uD + vD^c$.

This leads us to study the restriction to $S^1(\mathbb{C}) \subset P(\mathbb{C})$, where we can divide $\mathscr{T}^{pq}(\mathbb{V})$ by $(u + iv)^p (u - iv)^q$, giving

$$j_* \breve{\mathscr{A}}^\bullet_Y(\mathbb{V})|_{S^1(\mathbb{C})} \cong (j_* \mathscr{A}^*_Y(\mathbb{V})\mathscr{O}^{\mathrm{hol}}_{S^1}, t^{-1} \circledast D),$$

where (adapting Definition 2.8),

$$t^{-1} \circledast D := d^+ + t^{-1} \diamond \vartheta = \partial + \bar{\partial} + t^{-1}\theta + t\bar{\theta},$$

for $t \in \mathbb{C}^\times \cong S^1(\mathbb{C})$. There is a similar expression for $\breve{\mathscr{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})}$.

Now, as observed in [Sim1, end of §3], $t^{-1} \circledast D$ defines a holomorphic family $\mathscr{K}(\mathbb{V})$ of local systems on Y, parametrised by $S^1(\mathbb{C}) = \mathbb{C}^\times$. Beware that for non-unitary points $\lambda \in \mathbb{C}^\times$, the canonical metric is not pluriharmonic on the fibre $\mathscr{K}(\mathbb{V})_\lambda$, since $\lambda^{-1}\theta + \lambda\bar{\theta}$ is not Hermitian. The proof of Theorem 3.16 (essentially [Tim2, Proposition 1.7] and [Tim1, Theorem D.2(a)]) still adapts to verify conditions (1) and (2) from Proposition 3.10, replacing $\mathbb{V}$ with $\mathscr{K}(\mathbb{V})$, so that for instance

$$j_* \mathscr{K}(\mathbb{V}) \to J_0 \breve{\mathscr{T}}^\bullet(\mathbb{V})|_{S^1(\mathbb{C})}$$

is a quasi-isomorphism.

It remains to verify condition (3) from Proposition 3.10: we need to show that for all $a, b \geq 0$, the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant sheaf

$$\ker(\mathbb{H}^a(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^* \overline{\mathbb{H}^a(X, \mathrm{gr}^J_b \breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \to \mathrm{H}^a(X, \mathscr{H}^b(j_* \breve{\mathscr{A}}^\bullet_Y(\mathbb{V})))|_{S^1(\mathbb{C})})$$

is a finite locally free $\mathscr{O}_P^{\mathrm{hol}}$-module of slope $a + 2b$.

Arguing as in the proof of Theorem 3.16, we may apply a twist to reduce to the case $b = 0$ (replacing X with $D^{(b)}$ for the higher cases), so we wish to show that

$$\mathscr{E}^a := \ker(\mathbb{H}^a(X, J_0 \breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \oplus \sigma^* \overline{\mathbb{H}^a(X, J_0 \breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})}} \to \mathrm{H}^a(X, j_* \mathscr{K}(\mathbb{V}))|_{S^1(\mathbb{C})})$$

is a holomorphic vector bundle on $P(\mathbb{C})$ of slope a.

We do this by considering the graded sheaf $\mathscr{L}^*_{(2)}(\mathbb{V})$ of L^2-integrable distributions from Definition 3.15, and observe that [Tim1, Proposition D.4] adapts to show that

$$j_* \mathscr{K}(\mathbb{V}) \to (\mathscr{L}^*_{(2)}(\mathbb{V}) \mathscr{O}^{\mathrm{hol}}_{S^1}, t^{-1} \circledast D)$$

is a quasi-isomorphism on $X \times S^1(\mathbb{C})$.

On restricting to $\tilde{P}(\mathbb{C}) \subset P(\mathbb{C})$, Definition 3.5 gives the co-ordinate t on $\tilde{P}(\mathbb{C})$ as $t = \frac{u+iv}{u-iv}$, and dividing $\mathscr{T}^n(\mathbb{V})$ by $(u - iv)^n$ gives an isomorphism

$$\breve{\mathscr{T}}^\bullet(\mathbb{V})|_{\tilde{P}(\mathbb{C})} \cong (\mathscr{A}^*_X(\mathbb{V})\langle D \rangle \mathscr{O}^{\mathrm{hol}}_{\tilde{P}}, tD' + D''),$$

and similarly for $j_* \breve{\mathscr{A}}^\bullet_Y(\mathbb{V})|_{\tilde{P}(\mathbb{C})}$

Thus we also wish to show that

$$J_0 \breve{\mathscr{T}}^\bullet(\mathbb{V}))|_{\tilde{P}(\mathbb{C})} \to (\mathscr{L}^*_{(2)}(\mathbb{V}) \mathscr{O}^{\mathrm{hol}}_{\tilde{P}}, tD' + D'')$$

is a quasi-isomorphism. Condition (1) from Proposition 3.10 combines with the quasi-isomorphism above to show that we have a quasi-isomorphism on $S^1(\mathbb{C}) \subset \tilde{P}(\mathbb{C})$, so cohomology of the quotient is supported on $0 \in \tilde{P}(\mathbb{C})$. Studying the fibre over this point, it thus suffices to show that

$$(J_0 \mathscr{T}(\mathbb{V})), D'') \to (\mathscr{L}^*_{(2)}(\mathbb{V}) \otimes \mathbb{C}, D'')$$

is a quasi-isomorphism, which also follows by adapting [Tim1, Proposition D.4].

Combining the quasi-isomorphisms above gives an isomorphism

$$\mathscr{E}^a \cong \mathscr{H}^a(\breve{L}^\bullet_{(2)}(X, \mathbb{V})),$$

and inclusion of harmonic forms $\mathcal{H}^a(X, \mathbb{V}) \hookrightarrow L^a_{(2)}(X, \mathbb{V})$ gives a map

$$\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathscr{O}^{\mathrm{hol}}_P(a) \to \mathscr{H}^a(\breve{L}^\bullet_{(2)}(X, \mathbb{V})).$$

The Green's operator G behaves well in holomorphic families, so gives a decomposition

$$\breve{L}^a_{(2)}(X, \mathbb{V}) = \mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathscr{O}^{\mathrm{hol}}_P(a) \oplus \Delta \breve{L}^a_{(2)}(X, \mathbb{V}),$$

making use of finite-dimensionality of $\mathcal{H}^a(X, \mathbb{V})$ to give the isomorphism $\mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathscr{O}^{\mathrm{hol}}_P(a) \cong \ker \Delta \cap \breve{L}^a_{(2)}(X, \mathbb{V})$.

Since these expressions are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, it suffices to work on $\tilde{P}(\mathbb{C})$. Dividing $\mathscr{T}^n(\mathbb{V})$ by $(u - iv)^n$ gives

$$\breve{L}^\bullet_{(2)}(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} \cong (L^*_{(2)}(X, \mathbb{V}) \mathscr{O}^{\mathrm{hol}}_{\tilde{P}}, tD' + D'').$$

Now, since $D'(D'')^* + (D'')^* D' = 0$, we can write

$$\frac{1}{2}\Delta = (tD' + D'')(D'')^* + (D'')^*(tD' + D''),$$

giving us a direct sum decomposition

$$\breve{L}^a_{(2)}(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} = \mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathscr{O}^{\mathrm{hol}}_{\tilde{P}} \oplus (tD' + D'') \breve{L}^a_{(2)}(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})} \oplus (D'')^* \breve{L}^n_{(2)}(X, \mathbb{V})|_{\tilde{P}(\mathbb{C})},$$

with the principle of two types (as in [Sim2] Lemmas 2.1 and 2.2) showing that $(tD' + D'')\colon \operatorname{Im}((D'')^*) \to \operatorname{Im}(tD' + D'')$ is an isomorphism.

We have therefore shown that $\mathscr{E}^a \cong \mathcal{H}^a(X, \mathbb{V}) \otimes_{\mathbb{R}} \mathscr{O}_P^{\mathrm{hol}}(a)$, which is indeed of slope a. □

Proposition 3.20. *Assume that a Zariski-dense representation $\pi_1(Y, y) \to R(\mathbb{R})$ has unitary monodromy around the local components of D, and that the discrete S^1-action on ${}^{\nu}\varpi_1(Y, y)^{\mathrm{red}}$ descends to R. Then there are natural $(S^1)^\delta$-actions on $(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ and $\underline{\mathrm{gr}}(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$, compatible with the opposedness isomorphism, and with the action of $-1 \in S^1$ coinciding with that of $-1 \in \mathbb{G}_m$.*

Proof. We just observe that the construction $\mathscr{T}^*(\mathbb{V}) = \mathscr{A}^*_X(\mathbb{V})\langle D\rangle$ of Theorem 3.19 satisfies the conditions of Proposition 3.11, being closed under the $\circledast$-action of $\mathbb{C}^\times$. □

3.4. **Singular and simplicial varieties.** Fix a smooth proper simplicial complex variety $X_\bullet$, and a simplicial divisor $D_\bullet \subset X_\bullet$ with normal crossings. Set $Y_\bullet := X_\bullet - D_\bullet$, with a point $y \in Y_0$, and write $j : Y_\bullet \to X_\bullet$ for the embedding. Note that Proposition 2.24 shows that for any separated complex scheme Y of finite type, there exists such a simplicial variety $Y_\bullet$ with an augmentation $a\colon Y_\bullet \to Y$ for which $|Y_\bullet| \to Y$ is a weak equivalence.

Theorem 3.21. *Take $\rho\colon \pi_1(|Y_\bullet|, y) \to R(\mathbb{R})$ Zariski-dense with R pro-reductive, and assume that for every local system $\mathbb{V}$ on $|Y_\bullet|$ corresponding to an R-representation, the local system $a_0^{-1}\mathbb{V}$ on Y_0 is semisimple, with unitary monodromy around the local components of D_0. Then there is a canonical non-positively weighted mixed twistor structure $(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ on $(|Y_\bullet|, y)^{R,\mathrm{Mal}}$, in the sense of Definition 1.24.*

The associated split MTS is given by

$$\underline{\mathrm{gr}}(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \simeq \operatorname{Spec} \mathrm{Th}\,(\bigoplus_{p,q} \mathrm{H}^p(X_\bullet, a^{-1}\mathbf{R}^q j_*\mathbb{O}(R))[-p-q], d_2),$$

with $\mathrm{H}^p(X_n, \mathbf{R}^q j_* a_n^{-1}\mathbb{O}(R))$ of weight $p + 2q$. Here, $\mathrm{H}^p(X_\bullet, a^{-1}\mathbb{V})$ denotes the cosimplicial vector space $n \mapsto \mathrm{H}^p(X_n, a_n^{-1}\mathbb{V})$, and Th is the Thom-Whitney functor of Definition 2.29.

Proof. Our first observation is that the pullback of a holomorphic pluriharmonic metric is holomorphic, so for any local system $\mathbb{V}$ corresponding to an R-representation, the local system $a_n^{-1}\mathbb{V}$ on Y_n is semisimple for all n, with unitary monodromy around the local components of D_n. We may therefore form objects

$$(Y_n, (\sigma_0)^n y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m),$$

and $\underline{\mathrm{gr}}(Y_n, (\sigma_0)^n y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in dg_{\mathbb{Z}}\mathrm{Aff}(R)_*(\mathrm{Mat}_1)$ as in the proof of Theorem 3.19, together with opposedness quasi-isomorphisms.

These constructions are functorial, giving cosimplicial DGAs

$$\mathscr{O}(Y_\bullet, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \in cDG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1 \times C^*}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m),$$

and $\mathscr{O}(\underline{\mathrm{gr}}(Y_\bullet, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \in cDG_{\mathbb{Z}}\mathrm{Aff}(R)_*(\mathrm{Mat}_1)$. We now apply the Thom-Whitney functor, giving an algebraic MTS with $\underline{\mathrm{gr}}(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ as above, and

$$\mathscr{O}(|Y_\bullet|, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} := \mathrm{Th}\,(\mathscr{O}(Y_\bullet, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}).$$

Taking the fibre over $(1,1) \in \mathbb{A}^1 \times C^*$ gives $\mathrm{Th}\,(\mathscr{O}(Y_\bullet, y)^{R,\mathrm{Mal}})$, which is quasi-isomorphic to $\mathscr{O}(|Y_\bullet|, y)^{R,\mathrm{Mal}}$, by [Pri4, Lemma 9.11]. □

Theorem 3.22. *Take $\rho\colon \pi_1(|Y_\bullet|, y) \to R(\mathbb{R})$ Zariski-dense with R pro-reductive, and assume that for every local system $\mathbb{V}$ on $|Y_\bullet|$ corresponding to an R-representation, the local system $a_0^{-1}\mathbb{V}$ underlies a variation of Hodge structure with unitary monodromy around the local components of D_0. Then there is a canonical non-positively weighted mixed Hodge*

structure $(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$ *on* $(Y,y)^{R,\mathrm{Mal}}$*, in the sense of Definition 1.23. The associated split MTS is given by*

$$\underline{\mathrm{gr}}(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \simeq \mathrm{Spec}\,\mathrm{Th}\,(\bigoplus_{p,q} \mathrm{H}^p(X_\bullet, \mathbf{R}^q j_* a^{-1}\mathbb{O}(R))[-p-q], d_2),$$

with $\mathrm{H}^p(X_n, \mathbf{R}^q j_* a_n^{-1}\mathbb{O}(R))$ *a pure ind-Hodge structure of weight* $p+2q$*.*

Proof. The proof of Theorem 3.21 carries over, replacing Theorem 3.19 with Theorem 3.16, and observing that variations of Hodge structure are preserved by pullback. □

Definition 3.23. Define ${}^\nu\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}$ to be the quotient of $\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}$ characterised as follows. Representations of ${}^\nu\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}$ correspond to local systems $\mathbb{V}$ on $|Y_\bullet|$ for which $a_0^{-1}\mathbb{V}$ is a semisimple local system on Y_0 whose monodromy around local components of D_0 has unitary eigenvalues.

Proposition 3.24. *There is a discrete action of the circle group* S^1 *on* ${}^\nu\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}$*, such that the composition* $S^1 \times \pi_1(|Y_\bullet|,y) \to {}^\nu\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}$ *is continuous. We denote this last map by* $\sqrt{h} : \pi_1(|Y_\bullet|,y) \to {}^\nu\varpi_1(|Y_\bullet|,y)^{\mathrm{norm}}((S^1)^{\mathrm{cts}})$*.*

Proof. The proof of [Pri4, Proposition 9.8] carries over to the quasi-projective case. □

Proposition 3.25. *Take a pro-reductive* S^1*-equivariant quotient* R *of* ${}^\nu\varpi_1(|Y_\bullet|,x)^{\mathrm{norm}}$*, and assume that for every local system* $\mathbb{V}$ *on* $|Y_\bullet|$ *corresponding to an* R*-representation, the local system* $a_0^{-1}\mathbb{V}$ *has unitary monodromy around the local components of* D_0*. Then there are natural* $(S^1)^\delta$*-actions on* $(|Y_\bullet|,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ *and* $\underline{\mathrm{gr}}(|Y_\bullet|,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$*, compatible with the opposedness isomorphism, and with the action of* $-1 \in S^1$ *coinciding with that of* $-1 \in \mathbb{G}_m$*.*

Proof. This just follows from the observation that the S^1-action of Proposition 3.20 is functorial, hence compatible with the construction of Theorem 3.21. □

3.5. **More general monodromy.** It is natural to ask whether the hypotheses of Theorems 3.16 and 3.19 are optimal, or whether algebraic mixed Hodge and mixed twistor structures can be defined more widely. The analogous results to Theorem 3.16 for ℓ-adic pro-algebraic homotopy types in [Pri5] holds in full generality (i.e. for any Galois-equivariant quotient R of $\varpi_1(Y,y)^{\mathrm{red}}$). However the proofs of Theorems 3.16 and 3.19 clearly do not extend to non-unitary monodromy, since if θ is not holomorphic, then $\bar{\theta}$ does not act on $\mathscr{A}^*_X(\mathbb{V})\langle D\rangle$. Thus any proof adapting those theorems would have to take some modification of $\mathscr{A}^*_X(\mathbb{V})\langle D\rangle$ closed under the operator $\bar{\theta}$.

A serious obstruction to considering non-semisimple monodromy around the divisor is that the principle of two types plays a crucial rôle in the proofs of Theorems 3.16 and 3.19, and for quasi-projective varieties this is only proved for L^2 cohomology. The map $\mathrm{H}^*(X, j_*\mathbb{V}) \to \mathrm{H}^*_{(2)}(X,\mathbb{V})$ is only an isomorphism either for X a curve or for semisimple monodromy, so $\mathscr{L}^\bullet_{(2)}(\mathbb{V})$ will no longer have the properties we require. There is not even any prospect of modifying the filtrations in Propositions 3.2 or 3.10 so that $J_0\mathrm{H}^*(Y,\mathbb{V}) := \mathrm{H}^*_{(2)}(X,\mathbb{V})$, because L^2 cohomology does not carry a cup product *a priori* (and nor does intersection cohomology). This means that there is little prospect of applying the decomposition theorems of [Sab] and [Moc2], except possibly in the case of curves.

If the groups $\mathrm{H}^n(X, j_*\mathbb{V})$ all carry natural MTS or MHS, then the other terms in the Leray spectral sequence should inherit MHS or MTS via the isomorphisms

$$\mathrm{H}^n(X, \mathbf{R}^m j_*\mathbb{V}) \cong \mathrm{H}^n(X, \mathbf{R}^m j_*\mathbb{R} \otimes (j_*\mathbb{V}^\vee)^\vee) \cong \mathrm{H}^n(D^{(m)}, j_{m*}j_m^{-1}\nu_m^{-1}(j_*\mathbb{V}^\vee)^\vee \otimes \varepsilon^m),$$

for $j_m\colon (D^m - D^{m+1}) \to D^{(m)}$ the canonical open immersion. Note that $j_m^{-1}\nu_m^{-1}(j_*\mathbb{V}^\vee)^\vee$ is a local system on $D^m - D^{m+1}$ — this will hopefully inherit a tame pluriharmonic metric from $\mathbb{V}$ by taking residues.

It is worth noting that even for non-semisimple monodromy, the weight filtration on homotopy types should just be the one associated to the Leray spectral sequence. Although the monodromy filtration is often involved in such weight calculations, [Del3] shows that for $\mathbb{V}$ pure of weight 0 on Y, we still expect $j_*\mathbb{V}$ to be pure of weight 0 on X. It is only at generic (not closed) points of X that the monodromy filtration affects purity.

Adapting L^2 techniques to the case of non-semisimple monodromy around the divisor would have to involve some complex of Fréchet spaces to replace $L^\bullet_{(2)}(X,\mathbb{V})$, with the properties that it calculates $\mathrm{H}^*(X, j_*\mathbb{V})$ and is still amenable to Hodge theory. When monodromy around D is trivial, a suitable complex is $A^\bullet(X, j_*\mathbb{V})$, since $j_*\mathbb{V}$ is a local system. In general, one possibility is a modification of Foth's complex $\mathscr{B}^\bullet(\mathbb{V})$ from [Fot], based on bounded forms. Another possibility might be the complex given by $\bigcap_{p\in(0,\infty)} L^\bullet_{(p)}(X,\mathbb{V})$, i.e. the complex consisting of distributions which are L^p for all $p<\infty$. Beware that these are not the same as bounded forms — p-norms are all defined, but the limit $\lim_{p\to\infty}\|f\|_p$ might be infinite (as happens for $\log|\log|z||$).

Rather than using Fréchet space techniques directly, another approach to defining the MHS or MTS we need (including for $\mathbb{V}$ with non-semisimple monodromy) might be via Saito's mixed Hodge modules or Sabbah's mixed twistor modules. Since $\mathrm{H}^n(X, j_*\mathbb{V}) \cong \mathrm{IH}^n(X,\mathbb{V})$ for curves X, fibring by families of curves then opens the possibility of putting MHS or MTS on $\mathrm{H}^n(X, j_*\mathbb{V})$ for general X. Again, the main difficulty would lie in defining the cup products needed to construct DGAs.

4. Canonical splittings

4.1. Splittings of mixed Hodge structures.

Definition 4.1. For S as in Definition 1.2, define an S-action on SL_2 by
$$(\lambda, A)\mapsto \begin{pmatrix} 1 & 0\\ 0 & \lambda\bar{\lambda}\end{pmatrix} A \begin{pmatrix} \Re\lambda & \Im\lambda\\ -\Im\lambda & \Re\lambda\end{pmatrix}^{-1} = \begin{pmatrix} \lambda\bar{\lambda} & 0\\ 0 & 1\end{pmatrix}^{-1} A \begin{pmatrix} \Re\lambda & -\Im\lambda\\ \Im\lambda & \Re\lambda\end{pmatrix},$$
for any real algebra B, any $\lambda\in(B\otimes_{\mathbb{R}}\mathbb{C})^\times$ and any $A\in\mathrm{SL}_2(B)$.

Let $\mathrm{row}_1:\mathrm{SL}_2\to C^*$ be the S-equivariant map given by projection onto the first row.

Taking co-ordinates $\begin{pmatrix} u & v\\ x & y\end{pmatrix}$ for SL_2, we have $O(\mathrm{SL}_2)=\mathbb{R}[u,v,x,y]/(uy-vx-1)$. If we set $w=u+iv$, $\bar{w}=u-iv$ (as in Remark 1.3), $z=x+iy$ and $\bar{z}=x-iy$, then for the S-action we have w of type $(-1,0)$, $\bar{w}$ of type $(0,-1)$, z of type $(0,1)$ and $\bar{z}$ of type $(1,0)$.

Lemma 4.2. *The S-equivariant quasi-coherent ringed sheaf $\mathrm{row}_{1*}\mathscr{O}_{\mathrm{SL}_2}$ on C^* is flat, corresponding under Lemma 1.5 to the real algebra*
$$\mathcal{S}:=\mathbb{R}[x],$$
with filtration $F^p(\mathcal{S}\otimes\mathbb{C})=(x-i)^p\mathbb{C}[x]$.

Proof. This is [Pri4, Lemma 1.18] . □

Definition 4.3. Define the S-equivariant derivation $N: O(\mathrm{SL}_2)\to O(\mathrm{SL}_2)(-1)$ by $Nx=u, Ny=v, Nu=Nv=0$. Note that this is equivalent to the $O(\mathrm{SL}_2)$-linear isomorphism $\Omega(\mathrm{SL}_2/C)\to O(\mathrm{SL}_2)(-1)$ given by $dx\mapsto u$, $dy\mapsto v$. Observe that $\ker N = O(C)$.

Definition 4.4. Define MHS to be the category of finite-dimensional mixed Hodge structures.

Write $\mathrm{row}_2:\mathrm{SL}_2\to\mathbb{A}^2$ for projection onto the second row, so $\mathrm{row}_2^\sharp O(\mathbb{A}^2)$ is a subring of $O(\mathrm{SL}_2)$. This subring is equivariant for the S-action on SL_2 from Definition 4.1.

Definition 4.5. Define SHS (resp. ind(SHS)) to be the category of pairs (V,β), where V is a finite-dimensional S-representation (resp. an S-representation) in real vector spaces and $\beta: V\to V\otimes\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ is S-equivariant. A morphism $(V,\beta)\to(V',\beta')$ is an S-equivariant map $f:V\to V'$ with $\beta'\circ f=(f\otimes\mathrm{id})\circ\beta$.

Definition 4.6. Given $(V,\beta) \in \mathrm{SHS}$, observe that taking duals gives rise to a map $\beta^\vee : V^\vee \to V^\vee \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$. Then define the dual in SHS by $(V,\beta)^\vee := (V^\vee, \beta^\vee)$.

Likewise, we define the tensor product $(U,\alpha)\otimes(V,\beta) := (U\otimes V, \alpha\otimes \mathrm{id} + \mathrm{id}\otimes\beta)$.

Observe that for $(V,\beta),(V',\beta') \in \mathrm{SHS}$,

$$\mathrm{Hom}_{\mathrm{SHS}}((V,\beta),(V',\beta')) \cong \mathrm{Hom}_{\mathrm{SHS}}((\mathbb{R},0),(V,\beta)^\vee \otimes (V',\beta')).$$

Lemma 4.7. *A (commutative) algebra (A,δ) in* $\mathrm{ind}(\mathrm{SHS})$ *consists of an S-equivariant (commutative) algebra A, together with an S-equivariant derivation $\delta : A \to A \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$.*

Proof. We need to endow $(A,\delta) \in \mathrm{SHS}$ with a unit $(\mathbb{R},0) \to (A,\delta)$, which is the same as a unit $1 \in A$, and with a (commutative) associative multiplication

$$\mu\colon\ (A,\delta)\otimes(A,\delta) \to (A,\delta).$$

Substituting for $\otimes$, this becomes $\mu\colon\ (A\otimes A, \delta\otimes\mathrm{id}+\mathrm{id}\otimes\delta) \to (A,\delta)$, so μ is a (commutative) associative multiplication on A, and for $a,b \in A$, we must have $\delta(a,b) = a\delta(b) + b\delta(a)$. □

Theorem 4.8. *The categories* MHS *and* SHS *are equivalent. This equivalence is additive, and compatible with tensor products and duals.*

Proof. Given $(V,\beta) \in \mathrm{SHS}$ as above, define a weight filtration on V by $W_rV = \bigoplus_{i\le r} \mathcal{W}_iV$, where $\mathcal{W}_*$ is the weight decomposition associated to the S-action (as in Definition 1.4). Since β is S-equivariant and $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ is of strictly positive weights, we have

$$\beta : W_rV \to (W_{r-1}V)\otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1).$$

Thus β gives rise to an S-equivariant map $V \to V\otimes O(\mathrm{SL}_2)(-1)$ for which $\beta(W_rV) \subset (W_{r-1}V)\otimes O(\mathrm{SL}_2)(-1)$ for all r. In particular, $(W_rV, \beta|_{W_rV}) \in \mathrm{SHS}$ for all r.

We now form $V\otimes O(\mathrm{SL}_2)$, then look at the S-equivariant derivation $N_\beta : V\otimes O(\mathrm{SL}_2) \to V\otimes O(\mathrm{SL}_2)(-1)$ given by $N_\beta = \mathrm{id}\otimes N + \beta\otimes\mathrm{id}$. Since $\ker N = O(C)$, this map is $O(C)$-linear; by Lemma 4.2, it corresponds under Lemma 1.5 to a real derivation

$$N_\beta : V\otimes\mathcal{S} \to V(-1)\otimes\mathcal{S}$$

such that $N_\beta \otimes_{\mathbb{R}} \mathbb{C}$ preserves Hodge filtrations F. The previous paragraph shows that $N_\beta((W_rV)\otimes\mathcal{S}) \subset (W_rV)(-1)\otimes\mathcal{S}$, with

$$\mathrm{gr}^W N_\beta = (\mathrm{id}\otimes N) : (\mathrm{gr}^W V)\otimes\mathcal{S} \to (\mathrm{gr}^W V)(-1)\otimes\mathcal{S}.$$

Therefore $M(V,\beta) := \ker(N_\beta) \subset V\otimes\mathcal{S}$ is a real vector space, equipped with an increasing filtration W, and a decreasing filtration F on $M(V,\beta)\otimes\mathbb{C}$. We need to show that $M(V,\beta)$ is a mixed Hodge structure.

Since $N : \mathcal{S} \to \mathcal{S}(-1)$ is surjective, the observation above that $\mathrm{gr}^W N_\beta = (\mathrm{id}\otimes N)$ implies that N_β must also be surjective (as the filtration W is bounded), so

$$0 \to M(V,\beta) \to V\otimes\mathcal{S} \xrightarrow{N_\beta} V(-1)\otimes\mathcal{S} \to 0$$

is a exact sequence; this implies that the functor M is exact.

Since $\mathrm{gr}_r^W(V,\beta) = (\mathcal{W}_rV, 0)$, we get that $M(\mathrm{gr}_r^W(V,\beta)) = \mathcal{W}_rV$. As M is exact, $\mathrm{gr}_r^W M(V,\beta) = M(\mathrm{gr}_r^W(V,\beta))$, so we have shown that $\mathrm{gr}_r^W M((V,\beta))$ is a pure weight r Hodge structure, and hence that $M(V,\beta) \in \mathrm{MHS}$. Thus we have an exact functor

$$M : \mathrm{SHS} \to \mathrm{MHS};$$

it is straightforward to check that this is compatible with tensor products and duals.

We need to check that M is an equivalence of categories. First, observe that for any S-representation V, we have an object $(V,0) \in \mathrm{SHS}$ with $M(V) = V$.

Write

$$\mathrm{Ext}^1_{\mathrm{SHS}}((U,\alpha),(V,\beta)) := \mathrm{coker}\,(\beta_* - \alpha^*\colon \mathrm{Hom}_S(U,V) \xrightarrow{\beta_*-\alpha^*} \mathrm{Hom}_S(U, V\otimes O(C))).$$

This gives a an exact sequence

$$\begin{aligned} 0 \to &\mathrm{Hom}_{\mathrm{SHS}}((U,\alpha),(V,\beta)) \to \mathrm{Hom}_S(U,V) \xrightarrow{\beta_*-\alpha^*} \mathrm{Hom}_S(U,V\otimes O(C)) \\ \to &\mathrm{Ext}^1_{\mathrm{SHS}}((U,\alpha),(V,\beta)) \to 0. \end{aligned}$$

Note that $\mathrm{Ext}^1_{\mathrm{SHS}}((U,\alpha),(V,\beta))$ does indeed parametrise extensions of (U,α) by (V,β): given an exact sequence

$$0 \to (V,\beta) \to (W,\gamma) \to (U,\alpha) \to 0,$$

we may choose an S-equivariant section s of $W \twoheadrightarrow U$, so $W \cong U \oplus V$. The obstruction to this being a morphism in SHS is $o(s) := s^*\gamma - \alpha \in \mathrm{Hom}_S(U, V\otimes O(C))$, and another choice of section differs from s by some $f \in \mathrm{Hom}_S(U,V)$, with $o(s+f) = o(s) + \beta_* f - \alpha^* f$.

Write $\mathbf{R}^i\Gamma_{\mathrm{SHS}}(V,\beta) := \mathrm{Ext}^i((\mathbb{R},0),(V,\beta))$ for $i=0,1$, noting that

$$\mathrm{Ext}^i_{\mathrm{SHS}}((U,\alpha),(V,\beta))) = \mathbf{R}^i\Gamma_{\mathrm{SHS}}((V,\beta)\otimes(U,\alpha)^\vee).$$

We thus have morphisms

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{\mathrm{SHS}}(V,\beta) \arrow[r] \arrow[d] & V^S \arrow[r, "\beta"] \arrow[d] & (V\otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^S \arrow[r] \arrow[d] & \mathbf{R}^1\Gamma_{\mathrm{SHS}}(V,\beta) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma_{\mathcal{H}}M(V,\beta) \arrow[r] & (V\otimes O(\mathrm{SL}_2))^S \arrow[r, "\beta+N"] & (V\otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^S \arrow[r] & \mathbf{R}^1\Gamma_{\mathcal{H}}M(V,\beta) \arrow[r] & 0 \end{tikzcd}$$

of exact sequences, making use of the calculations of [Pri4, §1.3.1]. For any short exact sequence in SHS, the morphisms $\rho^i\colon \mathbf{R}^i\Gamma_{\mathrm{SHS}}(V,\beta) \to \mathbf{R}^i\Gamma_{\mathcal{H}}M(V,\beta)$ are thus compatible with the long exact sequences of cohomology.

The crucial observation on which the construction hinges is that the map $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1) \to \mathrm{coker}\,(N : O(\mathrm{SL}_2) \to O(\mathrm{SL}_2)(-1))$ is an isomorphism, making $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ a section for $O(\mathrm{SL}_2)(-1) \twoheadrightarrow \mathrm{H}^1(C^*, \mathscr{O}_{C^*})$. This implies that when $\beta = 0$, the maps ρ^i are isomorphisms. Since each object $(V,\beta) \in \mathrm{SHS}$ is an Artinian extension of S-representations, we deduce that the maps ρ^i must be isomorphisms for all such objects.

Taking $i=1$ gives that $\mathrm{Ext}^1_{\mathrm{SHS}}((U,\alpha),(V,\beta)) \to \mathrm{Ext}^1_{\mathcal{H}}(M(U,\alpha), M(V,\beta))$ is an isomorphism; we deduce that every extension in MHS lifts uniquely to an extension in SHS, so $M : \mathrm{SHS} \to \mathrm{MHS}$ is essentially surjective. Taking $i=0$ shows that M is full and faithful. □

Remark 4.9. Note that the Tannakian fundamental group (in the sense of [DMOS]) of the category SHS is

$$\Pi(\mathrm{SHS}) = S \ltimes \mathrm{Fr}(\mathcal{W}_{>0}(\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)))^\vee,$$

where $\mathrm{Fr}(V)$ denotes the free pro-unipotent group generated by the pro-finite-dimensional vector space V. In other words, SHS is canonically equivalent to the category of finite-dimensional $\Pi(\mathrm{SHS})$-representations. Likewise, ind(SHS) is equivalent to the category of all $\Pi(\mathrm{SHS})$-representations.

The categories SHS and MHS both have vector space-valued forgetful functors. Tannakian formalism shows that the functor $\mathrm{SHS} \to \mathrm{MHS}$, together with a choice of natural isomorphism between the respective forgetful functors, gives a morphism $\Pi(\mathrm{MHS}) \to \Pi(\mathrm{SHS})$. The choice of natural isomorphism amounts to choosing a Levi decomposition for $\Pi(\mathrm{MHS})$, or equivalently a functorial isomorphism $V \cong \mathrm{gr}^W V$ of vector spaces for $V \in \mathrm{MHS}$.

A canonical choice b_0 of such an isomorphism is given by composing the embedding $b\colon M(V,\beta) \hookrightarrow V \otimes \mathcal{S}$ with the map $p_0\colon \mathcal{S} \to \mathbb{R}$ given by $x \mapsto 0$. This allows us to put a new MHS on V, with Hodge filtration $b_0(F)$ and the same weight filtration as V, so

$b_0\colon M(V,\beta) \to (V, W, b_0(F))$ is an isomorphism of MHS. To describe this new MHS, first observe that $\mathcal{S}(-1) \cong \Omega(\mathcal{S}/\mathbb{R}) = \mathcal{S}dx$, and that for $\beta : V \to V \otimes \Omega(\mathcal{S}/\mathbb{R})$, we get an isomorphism $\exp(-\int_0^x \beta)\colon V \to M(V,\beta)$, which is precisely b_0^{-1}.

Since the map $p_i\colon \mathcal{S} \to \mathbb{C}$ given by $x \mapsto i$ preserves F, it follows that the map

$$p_i \circ b_0^{-1} = \exp(-\int_0^i \beta)\colon V \to V \otimes \mathbb{C}$$

satisfies $\exp(-\int_0^i \beta)(b_0(F)) = F$, so the new MHS is

$$(V, W, b_0(F)) = (V, W, \exp(\int_0^i \beta)(F)).$$

Remark 4.10. In [Pri4, Proposition 1.25], it was shown that every mixed Hodge structure M admits a non-unique splitting $M \otimes \mathcal{S} \cong (\mathrm{gr}^W M) \otimes \mathcal{S}$, compatible with the filtrations. Theorem 4.8 is a refinement of that result, showing that such a splitting can be chosen canonically, by requiring that the image of $\mathrm{gr}^W M$ under the derivation $(\mathrm{id}_M \otimes N\colon M \otimes O(\mathrm{SL}_2) \to M \otimes O(\mathrm{SL}_2)(-1)$ lies in $\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$. This is because β is just the restriction of $\mathrm{id}_M \otimes N$ to $V := \mathrm{gr}^W M$.

This raises the question of which F-preserving maps $\beta : V \to V \otimes \Omega(\mathcal{S}/\mathbb{R})$ correspond to maps $V \to V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ (rather than just $V \to V \otimes O(\mathrm{SL}_2)(-1)$. Using the explicit description from the proof of [Pri4, Lemma 1.18], we see that this amounts to the restriction that

$$\beta(V_{\mathbb{C}}^{p,q}) \subset \sum_{a\geq 0, b\geq 0} V_{\mathbb{C}}^{p-a-1,q-b-1}(x-i)^a(x+i)^b dx.$$

Remark 4.11. In [Del4], Deligne established a characterisation of real MHS in terms of S-representations equipped with additional structure.

For any $\lambda \in \mathbb{C}$, we have a map $p_\lambda\colon \mathcal{S} \to \mathbb{C}$ given by $x \mapsto \lambda$, and $b_\lambda^{-1} := (p_\lambda \circ b)^{-1} = \exp(-\int_\lambda^x \beta)\colon V \to M(V,\beta)$. Comparing the filtrations $b_0(F)$ and $b_0(\bar{F})$ on V, we are led to consider

$$d := b_{-i} \circ b_i^{-1} = \exp(\int_{-i}^i \beta).$$

This maps V to V, and has the properties that $\bar{d} = d^{-1}$ and

$$(d - \mathrm{id})(V_{\mathbb{C}}^{pq}) \subset \bigoplus_{r<p, s<q} V_{\mathbb{C}}^{rs}.$$

This is precisely the data of an $\mathfrak{M}$-representation in the sense of [Del4, Proposition 2.1], so corresponds to a MHS. Explicitly, we first find the unique operator $d^{1/2}$ satisfying the properties for d above, then define the mixed Hodge structure $M(V,d)$ to have underlying vector space V, with the same weight filtration, and with $F^p M(V,d) := d^{1/2}(F^p V)$.

For our choice of d as above, we then have an isomorphism

$$a := d^{1/2} \circ b_i = d^{-1/2} \circ b_{-i}\colon M(V,\beta) \to V$$

of vector spaces. Since $b_i(F^p M(V,\beta)) = F^p V$, this means that $a(F^p M(V,\beta)) = F^p M(V,d)$, so a is an isomorphism of MHS.

We have therefore shown directly how our category SHS is equivalent to Deligne's category of $\mathfrak{M}$-representations by sending the pair (V,β) to $(V, \exp(\int_{-i}^i \beta))$. This also gives a canonical isomorphism $\mathfrak{M} \cong \Pi(\mathrm{SHS})$, once we specify the associated isomorphism $a \circ b_0^{-1}\colon V \to V$ on fibre functors. This isomorphism can be understood in terms of identifying the generating elements of [Del4, Construction 1.6] with explicit elements of $\mathrm{row}_2^\sharp(O(\mathbb{A}^2)(-1)) \otimes \mathbb{C}$.

For an explicit quasi-inverse functor from $\mathfrak{M}$-representations to SHS, take a pair (V, d). Since d is unipotent, $\delta := \log d \colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ is well-defined, and decomposes into types as $\delta = \sum_{a,b<0} \delta^{ab}$. We now just set

$$\beta := \sum_{a \geq 0, b \geq 0} \frac{\delta^{-a-1,-b-1}(x-i)^a(x+i)^b dx}{\int_{-i}^{i}(x-iy)^a(x+iy)^b dx},$$

for co-ordinates x, y on $\mathbb{A}^2$.

4.2. **Splittings of mixed twistor structures.** The following lemma ensures that a mixed twistor structure can be regarded as an Artinian extension of $\mathbb{G}_m$-representations.

Lemma 4.12. *If $\mathscr{E}$ and $\mathscr{F}$ are pure twistor structures of weights m and n respectively, then*

$$\mathrm{Hom}_{\mathrm{MTS}}(\mathscr{E}, \mathscr{F}) \cong \begin{cases} \mathrm{Hom}_{\mathbb{R}}(\mathscr{E}_1, \mathscr{F}_1) & m = n \\ 0 & m \neq n. \end{cases}$$

Proof. By hypothesis, $\mathscr{E} = \mathrm{gr}^W_m \mathscr{E}$ and $\mathscr{F} = \mathrm{gr}^W_n \mathscr{F}$. Thus we may assume that $\mathscr{E} = \mathscr{O}(m)$ and $\mathscr{F} = \mathscr{O}(n)$. Since homomorphisms must respect the weight filtration, we have

$$\mathrm{Hom}_{\mathrm{MTS}}(\mathscr{O}(m), \mathscr{O}(n)) = \mathrm{Hom}_{\mathbb{P}^1}(\mathscr{O}(m), W_m\mathscr{O}(n)),$$

which is 0 unless $m \geq n$. When $m \geq n$, we have $W_m\mathscr{O}(n) = \mathscr{O}(n)$, so

$$\mathrm{Hom}_{\mathrm{MTS}}(\mathscr{O}(m), \mathscr{O}(n)) = \Gamma(\mathbb{P}^1, \mathscr{O}(n-m)),$$

which is 0 for $m > n$ and $\mathbb{R}$ for $n = m$, as required. □

Definition 4.13. Define STS to be the category of pairs (V, β), where V is an $\mathbb{G}_m$-representation in real vector spaces and $\beta : V \to V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$ is $\mathbb{G}_m$-equivariant. A morphism $(V, \beta) \to (V', \beta')$ is a $\mathbb{G}_m$-equivariant map $f : V \to V'$ with $\beta' \circ f = (f \otimes \mathrm{id}) \circ \beta$.

Note that the only difference between Definitions 4.5 and 4.13 is that the latter replaces S with $\mathbb{G}_m$ throughout.

Definition 4.14. Given $(V, \beta) \in \mathrm{STS}$, observe that taking duals gives rise to a map $\beta^\vee : V^\vee \to V^\vee \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$. Then define the dual in STS by $(V, \beta)^\vee := (V^\vee, \beta^\vee)$.

Likewise, we define the tensor product by $(U, \alpha) \otimes (V, \beta) := (U \otimes V, \alpha \otimes \mathrm{id} + \mathrm{id} \otimes \beta)$.

Observe that for $(V, \beta), (V', \beta') \in \mathrm{STS}$,

$$\mathrm{Hom}_{\mathrm{STS}}((V, \beta), (V', \beta')) \cong \mathrm{Hom}_{\mathrm{STS}}((\mathbb{R}, 0), (V, \beta)^\vee \otimes (V', \beta')).$$

Theorem 4.15. *The categories* MTS *and* STS *are equivalent. This equivalence is additive, and compatible with tensor products and duals.*

Proof. As in the proof of Theorem 4.8, every object $(V, \beta) \in \mathrm{STS}$ inherits a weight filtration W from V, and β gives rise to a $\mathbb{G}_m$-equivariant map

$$N_\beta : V \otimes O(\mathrm{SL}_2) \to V \otimes O(\mathrm{SL}_2)(-1)$$

respecting the weight filtration on V, with $\mathrm{gr}^W N_\beta = (\mathrm{id} \otimes N)$.

For the projection $\mathrm{row}_1 : \mathrm{SL}_2 \to C^*$ of Definition 4.1, we then get a $\mathbb{G}_m$-equivariant map

$$\mathrm{row}_{1*}N_\beta : \mathrm{row}_{1*}(V \otimes \mathscr{O}_{\mathrm{SL}_2}) \to \mathrm{row}_{1*}(V \otimes \mathscr{O}_{\mathrm{SL}_2}(-1));$$

Then $\ker(\mathrm{row}_{1*}N_\beta)$ is a $\mathbb{G}_m$-equivariant vector bundle on C^*. Using the isomorphism $C \cong \mathbb{A}^2$ of Remark 1.3 and the projection $\pi : (\mathbb{A}^2 - \{0\}) \to \mathbb{P}^1$, this corresponds to a vector bundle $M(V, \beta) := (\pi_* \ker(\mathrm{row}_{1*}N_\beta))^{\mathbb{G}_m}$ on $\mathbb{P}^1$.

Now, $M(V, \beta)$ inherits a weight filtration W from V, and surjectivity of N_β implies that

$$0 \to \ker(\mathrm{row}_{1*}N_\beta) \to \mathrm{row}_{1*}(V \otimes \mathscr{O}_{\mathrm{SL}_2}) \to \mathrm{row}_{1*}(V \otimes \mathscr{O}_{\mathrm{SL}_2}(-1)) \to 0$$

is an exact sequence, so M is an exact functor. In particular, this gives $\mathrm{gr}_n^W M(V,\beta) = M(\mathcal{W}_n V, 0)$, which is just the vector bundle on $\mathbb{P}^1$ corresponding to the $\mathbb{G}_m$-equivariant vector bundle $(\mathcal{W}_n V)\otimes \mathscr{O}_{C^*}$ on C^*. Since $\mathcal{W}_n V$ has weight n for the $\mathbb{G}_m$-action, this means that $\mathrm{gr}_n^W M(V,\beta)$ has slope n, so we have defined an exact functor

$$M : \mathrm{STS} \to \mathrm{MTS},$$

which is clearly compatible with tensor products and duals.

If we define $\Gamma_{\mathrm{STS}}(V,\beta) := \ker(\beta\colon\ V \to V \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^{\mathbb{G}_m}$ and $\mathbf{R}^1\Gamma_{\mathrm{STS}}(V,\beta) := (\mathrm{coker}\,\beta)^{\mathbb{G}_m}$, then the proof of Theorem 4.8 gives us morphisms

$$\rho^i\colon\ \mathbf{R}^i\Gamma_{\mathrm{STS}}(V,\beta) \to W_0\mathrm{H}^i(\mathbb{P}^1, M(V,\beta))$$

for $i = 0,1$. These are automatically isomorphisms when $\beta = 0$, and the long exact sequences of cohomology then give that ρ^i is an isomorphism for all (V,β). We therefore have isomorphisms

$$\mathrm{Ext}^i_{\mathrm{STS}}((U,\alpha),(V,\beta)) \to W_0\mathrm{Ext}^i_{\mathbb{P}^1}(M(U,\alpha), M(V,\beta)),$$

and arguing as in Theorem 4.8, this shows that M is an equivalence of categories, using Lemma 4.12 in the pure case. □

Remark 4.16. Note that the Tannakian fundamental group (in the sense of [DMOS]) of the category STS is

$$\Pi(\mathrm{SHS}) = \mathbb{G}_m \ltimes \mathrm{Fr}(\mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1))^\vee,$$

where $\mathrm{Fr}(V)$ denotes the free pro-unipotent group generated by the pro-finite-dimensional vector space V.

The functor STS $\to$ MTS then gives a morphism $\Pi(\mathrm{MTS}) \to \Pi(\mathrm{STS})$, but this is not unique, since it depends on a choice of natural isomorphism between the fibre functors (at $1 \in C^*$) on MTS and on STS. This amounts to choosing a Levi decomposition for $\Pi(\mathrm{MTS})$, or equivalently a functorial isomorphism $\mathscr{E}_1 \cong \mathrm{gr}^W\mathscr{E}_1$ of vector spaces for $\mathscr{E} \in$ MHS. A canonical choice of such an isomorphism is to take the fibre at $I \in \mathrm{SL}_2$.

We can think of Theorem 4.15 as an analogue of [Del4] for real mixed twistor structures, in that for any MTS $\mathscr{E}$, it gives a canonical splitting of the weight filtration on $\mathscr{E}_1$, together with unique additional data required to recover $\mathscr{E}$.

5. SL_2 splittings of non-abelian MTS/MHS and strictification

5.1. **Simplicial structures.**

Definition 5.1. Let $s\mathrm{Cat}$ be the category of simplicially enriched small categories, which we will refer to as simplicial categories. Explicitly, an object $\mathcal{C} \in s\mathrm{Cat}$ consists of a set $\mathrm{Ob}\,\mathcal{C}$ of objects, together with $\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y) \in \mathbb{S}$ for all $x,y \in \mathrm{Ob}\,\mathbb{S}$, equipped with an associative composition law and identities.

Lemma 5.2. *For a reductive pro-algebraic monoid M and an M-representation A in DG algebras, there is a cofibrantly generated model structure on $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$, in which fibrations are surjections, and weak equivalences are quasi-isomorphisms.*

Proof. When M is a group, this is [Pri4, Lemma 3.38], but the same proof carries over to the monoid case. □

Definition 5.3. Given $B \in DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$ define $B^{\Delta^n} := B \otimes_{\mathbb{Q}} \Omega(|\Delta^n|)$, for $\Omega(|\Delta^n|)$ as in Definition 2.29. Make $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$ into a simplicial category by setting $\underline{\mathrm{Hom}}(B,B')$ to be the simplicial set

$$\underline{\mathrm{Hom}}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)}(B,C)_n := \mathrm{Hom}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)}(B, C^{\Delta^n}).$$

Beware that $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$ does not then satisfy the axioms of a simplicial model category from [GJ] Ch. II, because $\underline{\mathrm{Hom}}(-,B) : DG_{\mathbb{Z}}\mathrm{Alg}_A(M)^{\mathrm{opp}} \to \mathbb{S}$ does not have a left adjoint. However, $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$ is a simplicial model category in the weaker sense of [Qui].

Now, as in [Hov, §5], for any pair X, Y of objects in a model category $\mathcal{C}$, there is a derived function complex $\mathbf{R}\mathrm{Map}_{\mathcal{C}}(X,Y) \in \mathbb{S}$, defined up to weak equivalence. One construction is to take a cofibrant replacement $\tilde{X}$ for X and a fibrant resolution $\hat{Y}_\bullet$ for Y in the Reedy category of simplicial diagrams in $\mathcal{C}$, then to set

$$\mathbf{R}\mathrm{Map}_{\mathcal{C}}(X,Y)_n := \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \hat{Y}_n).$$

In fact, Dwyer and Kan showed in [DK] that $\mathbf{R}\mathrm{Map}_{\mathcal{C}}$ is completely determined by the weak equivalences in $\mathcal{C}$. In particular, $\pi_0\mathbf{R}\mathrm{Map}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y)$, where $\mathrm{Ho}(\mathcal{C})$ is the homotopy category of $\mathcal{C}$, given by formally inverting weak equivalences.

To see that $C^{\Delta^\bullet}$ is a Reedy fibrant simplicial resolution of C in $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$, note that the matching object $M_n C^{\Delta^\bullet}$ is given by

$$C \otimes M_n\Omega(|\Delta^\bullet|) = C \otimes \Omega(|\Delta^n|)/(t_0\cdots t_n, \sum_i t_0 \cdots t_{i-1}(dt_i)t_{i+1}\cdots t_n),$$

so the matching map $C^{\Delta^n} \to M_n C^{\Delta^\bullet}$ is a fibration (i.e. surjective).

Therefore for $\tilde{B} \to B$ a cofibrant replacement,

$$\mathbf{R}\mathrm{Map}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)}(B,C) \simeq \underline{\mathrm{Hom}}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)}(\tilde{B}, C).$$

Definition 5.4. Given an object $D \in DG_{\mathbb{Z}}\mathrm{Alg}_A(M)$, make the comma category $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)\downarrow D$ into a simplicial category by setting

$$\underline{\mathrm{Hom}}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)\downarrow D}(B,C)_n := \mathrm{Hom}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)}(B, C^{\Delta^n} \times_{D^{\Delta^n}} D).$$

Now, $C \to C^{\Delta^\bullet} \times_{D^{\Delta^\bullet}} D$ is a Reedy fibrant resolution of C in $DG_{\mathbb{Z}}\mathrm{Alg}_A(M)\downarrow D$ for every fibration $C \to D$. Thus for $\tilde{B} \to B$ a cofibrant replacement and $C \to \hat{C}$ a fibrant replacement,

$$\mathbf{R}\mathrm{Map}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)\downarrow D}(B,C) \simeq \underline{\mathrm{Hom}}_{DG_{\mathbb{Z}}\mathrm{Alg}_A(M)\downarrow A}(\tilde{B}, \hat{C}).$$

Definition 5.5. Given a simplicial category $\mathcal{C}$, recall from [Ber] that the category $\pi_0\mathcal{C}$ is defined to have the same objects as $\mathcal{C}$, with morphisms

$$\mathrm{Hom}_{\pi_0\mathcal{C}}(x,y) = \pi_0\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y).$$

A morphism in $\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)_0$ is said to be a homotopy equivalence if its image in $\pi_0\mathcal{C}$ is an isomorphism.

If the objects of a simplicial category $\mathcal{C}$ are the fibrant cofibrant objects of a model category $\mathcal{M}$, with $\underline{\mathrm{Hom}}_{\mathcal{C}} = \mathbf{R}\mathrm{Map}_{\mathcal{M}}$, then observe that homotopy equivalences in $\mathcal{C}$ are precisely weak equivalences in $\mathcal{M}$.

5.2. **Functors parametrising Hodge and twistor structures.** The DG algebra $O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon$, for ϵ of degree 1, is an algebra over $O(C) = \mathbb{R}[u,v]$, so we may consider the DG algebra $j^{-1}O(\mathrm{SL}_2) \xrightarrow{N\epsilon} j^{-1}O(\mathrm{SL}_2)(-1)\epsilon$ on C^*, for $j : C^* \to C$. This is an acyclic resolution of the structure sheaf $\mathscr{O}_{C^*}$, so

$$\mathbf{R}j_*\mathscr{O}_{C^*} \simeq j_*(j^{-1}O(\mathrm{SL}_2) \xrightarrow{N\epsilon} j^{-1}O(\mathrm{SL}_2)(-1)\epsilon) = (O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon),$$

regarded as an $O(C)$-algebra. This construction is moreover S-equivariant.

Definition 5.6. From now on, we will denote the DG algebra $O(\mathrm{SL}_2) \xrightarrow{N\epsilon} O(\mathrm{SL}_2)(-1)\epsilon$ by $\mathbf{R}O(C^*)$, thereby making a canonical choice of representative in the equivalence class $\mathbf{R}\Gamma(C^*, \mathscr{O}_{C^*})$. We also denote the sheaf $j^{-1}\mathbf{R}O(C^*)$ on C^* by $\mathbf{R}\mathscr{O}_{C^*}$, giving a canonical acyclic resolution of $\mathscr{O}_{C^*}$.

Proposition 5.7. *For any R' acting on C^* and any R'-equivariant algebra A, the functor $j^* : DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(R') \to DG_{\mathbb{Z}}\mathrm{Alg}_{\mathrm{Spec}\, A\times C^*}(R')$ induces an equivalence*

$$\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(R')) \to \mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{\mathrm{Spec}\, A\times C^*}(R')).$$

For any R'-representation B in A-algebras, this extends to an equivalence

$$\mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(R')\downarrow B\otimes\mathbf{R}O(C^*)) \to \mathrm{Ho}(DG_{\mathbb{Z}}\mathrm{Alg}_{\mathrm{Spec}\, A\times C^*}(R)\downarrow B\otimes\mathscr{O}_{C^*}.$$

Proof. This is a special case of [Pri4, Proposition 3.45]. □

Definition 5.8. For $A\in\mathrm{Alg}(\mathrm{Mat}_1)$, define $\mathcal{PT}(A)_*$ (resp. $\mathcal{PH}(A)_*$) to be the full simplicial subcategory of the category

$$DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(\mathrm{Mat}_1\times R\times\mathbb{G}_m)\downarrow A\otimes O(R)\otimes\mathbf{R}O(C^*)$$
$$(\text{resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(\mathrm{Mat}_1\times R\rtimes S)\downarrow A\otimes O(R)\otimes\mathbf{R}O(C^*))$$

on fibrant cofibrant objects. These define functors

$$\mathcal{PT}_*,\, \mathcal{PH}_*\colon\ DG_{\mathbb{Z}}\mathrm{Alg}(\mathrm{Mat}_1)\to s\mathrm{Cat}.$$

Remark 5.9. Since $\mathcal{PT}(A)_*$ and $\mathcal{PH}(A)_*$ are defined in terms of derived function complexes, it follows that a morphism in any of these categories is a homotopy equivalence (in the sense of Definition 5.5) if and only if it is weak equivalence in the associated model category, i.e. a quasi-isomorphism.

Remark 5.10. Let $\mathbb{R}[t]\in\mathrm{Alg}(\mathrm{Mat}_1)$ be given by setting t to be of weight 1. After applying Proposition 5.7 and taking fibrant cofibrant replacements, observe that a pointed algebraic non-abelian mixed twistor structure consists of

$$O(\underline{\mathrm{gr}}X_{\mathrm{MTS}})\in DG_{\mathbb{Z}}\mathrm{Alg}(R\times\mathrm{Mat}_1)\downarrow O(R),$$

together with an object $O(X_{\mathrm{MTS}})\in\mathcal{PT}_*(\mathbb{R}[t])$ and a weak equivalence

$$O(X_{\mathrm{MTS}})\otimes_{\mathbb{R}[t]}\mathbb{R}\to O(\underline{\mathrm{gr}}X_{\mathrm{MTS}})$$

in $\mathcal{PT}_*(\mathbb{R})$.

Likewise, a pointed algebraic non-abelian mixed Hodge structure consists of

$$O(\underline{\mathrm{gr}}X_{\mathrm{MHS}})\in DG_{\mathbb{Z}}\mathrm{Alg}(R\rtimes\bar{S})\downarrow O(R),$$

together with an object $O(X_{\mathrm{MHS}})\in\mathcal{PH}_*(\mathbb{R}[t])$, and a weak equivalence

$$O(X_{\mathrm{MHS}})\otimes_{\mathbb{R}[t]}\mathbb{R}\to O(\underline{\mathrm{gr}}X_{\mathrm{MHS}})$$

in $\mathcal{PH}_*(\mathbb{R})$.

5.3. **Deformations.**

5.3.1. *Quasi-presmoothness.* The following is [Pri3, Definition 2.22]:

Definition 5.11. Say that a morphism $F:\mathcal{A}\to\mathcal{B}$ in $s\mathrm{Cat}$ is is a 2-fibration if

(F1) for any objects a_1 and a_2 in $\mathcal{A}$, the map $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_1,a_2)\to\underline{\mathrm{Hom}}_{\mathcal{B}}(Fa_1,Fa_2)$ is a fibration of simplicial sets;

(F2) for any objects $a_1\in\mathcal{A}$, $b\in\mathcal{B}$, and any homotopy equivalence $e: Fa_1\to b$ in $\mathcal{B}$, there is an object $a_2\in\mathcal{C}$, a homotopy equivalence $d: a_1\to a_2$ in $\mathcal{C}$ and an isomorphism $\theta: Fa_2\to b$ such that $\theta\circ Fd=e$.

The following are adapted from [Pri3]:

Definition 5.12. Say that a functor $\mathcal{D} : \mathrm{Alg}(\mathrm{Mat}_1) \to s\mathrm{Cat}$ is formally 2-quasi-presmooth if for all square-zero extensions $A \to B$, the map

$$\mathcal{D}(A) \to \mathcal{D}(B)$$

is a 2-fibration.

Say that $\mathcal{D}$ is formally 2-quasi-presmooth if $\mathcal{D} \to \bullet$ is so.

Proposition 5.13. *The functors $\mathcal{PT}_*, \mathcal{PH}_* : \mathrm{Alg}(\mathrm{Mat}_1) \to s\mathrm{Cat}$ are formally 2-quasi-presmooth.*

Proof. Apart from the augmentation maps, this is essentially the same as [Pri3, Proposition 3.14], which proves the corresponding statements for the functor on algebras given by sending A to the simplicial category of cofibrant DG $(T \otimes A)$-algebras, for T cofibrant. The same proof carries over, the only change being to take $\mathrm{Mat}_1 \times R \times \mathbb{G}_m$-invariants (resp. $\mathrm{Mat}_1 \times R \rtimes S$-invariants) of the André-Quillen cohomology groups. We now sketch the argument.

Let $\mathcal{P}$ be $\mathcal{PT}_*$ (resp. $\mathcal{PH}_*$), and write S' for $\mathbb{G}_m$ (resp. S). Fix a square-zero extension $A \to B$ in $\mathrm{Alg}(\mathrm{Mat}_1)$. Thus an object $C \in \mathcal{P}(B)$ is a $\mathrm{Mat}_1 \times R \rtimes S'$-equivariant diagram $B \otimes \mathbf{R}O(C^*) \to C \to B \otimes O(R) \otimes \mathbf{R}O(C^*)$, with the first map a cofibration and the second a fibration. Since C cofibrant, the underlying graded algebra is smooth over $B \otimes \mathbf{R}O(C^*)$, so lifts essentially uniquely to give a smooth morphism $A^* \otimes \mathbf{R}O(C^*)^* \to \tilde{C}^*$ of graded algebras, with $\tilde{C}^* \otimes_A B \cong C^*$. As $A \otimes O(R) \otimes \mathbf{R}O(C^*) \to B \otimes O(R) \otimes \mathbf{R}O(C^*)$ is square-zero, smoothness of $\tilde{C}^*$ gives us a lift $\tilde{p} : \tilde{C}^* \to A^* \otimes O(R) \otimes \mathbf{R}O(C^*)^*$. Since $\mathrm{Mat}_1 \times R \rtimes S'$ is reductive, these maps can all be chosen equivariantly.

Now, choose some equivariant A-linear derivation δ on $\tilde{C}$ lifting d_C. The obstruction to lifting $c \in \mathcal{P}(B)$ to $\mathcal{P}(A)$ up to isomorphism is then the class

$$\begin{aligned}[(\delta^2, p \circ \delta - d \circ p)] \in &\mathrm{H}^2\mathrm{HOM}_C(\Omega(C/(B \otimes \mathbf{R}O(C^*))), I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{R}O(C^*)) \\ = &\mathrm{Ext}^2_C(\mathbb{L}_\bullet^{C/(B \otimes \mathbf{R}O(C^*))}, I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{R}O(C^*)).\end{aligned}$$

This is because any other choice of $(\delta, \tilde{p})$ amounts to adding the boundary of an element in $\mathrm{HOM}^1_C(\Omega(C/(B \otimes \mathbf{R}O(C^*))), I \otimes_B C \xrightarrow{p} I \otimes O(R) \otimes \mathbf{R}O(C^*))$.

The key observation now is that the cotangent complex is an invariant of the quasi-isomorphism class, so C lifts to $\mathcal{P}(A)$ up to isomorphism if and only if all quasi-isomorphic objects also lift. The treatment of morphisms is similar. Although augmentations are not addressed in [Pri3, Proposition 3.14], the same proof adapts. It is important to note that the André-Quillen characterisation of obstructions to lifting morphisms does not require the target to be cofibrant. □

5.3.2. *Strictification.*

Proposition 5.14. *Let $\mathcal{P} : \mathrm{Alg}(\mathrm{Mat}_1) \to s\mathrm{Cat}$ be one of the functors $\mathcal{PT}_*$ or $\mathcal{PH}_*$. Given an object E in $\mathcal{P}(\mathbb{R})$, an object P in $\mathcal{P}(\mathbb{R}[t])$, and a quasi-isomorphism*

$$f : P/tP \to E$$

in $\mathcal{P}(\mathbb{R})$, there is an object $M \in \mathcal{P}(\mathbb{R}[t])$, a quasi-isomorphism $g : P \to M$, and an isomorphism $\theta : M/tM \to E$ such that $\theta \circ \bar{g} = f$.

Proof. If we replace $\mathbb{R}[t]$ with $\mathbb{R}[t]/t^r$, then the statement holds immediately from Proposition 5.13 and the definition of formal 2-quasi-presmoothness, since the extension $\mathbb{R}[t]/t^r \to \mathbb{R}$ is nilpotent. Proceeding inductively, we get a system of objects $M_r \in \mathcal{P}(\mathbb{R}[t]/t^r)$, quasi-isomorphisms $g_r : P/t^rP \to M_r$ and isomorphisms $\phi_r : M_r/t^{r-1}M_r \to M_{r-1}$ with $M_0 = E$, $g_0 = f$ and $\phi_r \circ \bar{g}_r = g_{r-1}$.

We may therefore set M to be the inverse limit of the system

$$\ldots \xrightarrow{\phi_{r+1}} M_r \xrightarrow{\phi_r} M_{r-1} \xrightarrow{\phi_{r-1}} \ldots \xrightarrow{\phi_1} M_0 = E$$

in the category of Mat_1-representations. Explicitly, this says that the maps

$$\mathcal{W}_n M \to \varprojlim_r \mathcal{W}_n M/(t^r \mathcal{W}_{n-r} M)$$

are isomorphisms for all n. In particular, beware that the forgetful functor from Mat_1-representations to vector spaces does not preserve inverse limits.

Let $\mathcal{M}(A)$ be one of the model categories

$$DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(\mathrm{Mat}_1 \times R \times \mathbb{G}_m)\downarrow A \otimes O(R) \otimes \mathbf{R}O(C^*)$$
$$\text{or } DG_{\mathbb{Z}}\mathrm{Alg}_{A\otimes\mathbf{R}O(C^*)}(\mathrm{Mat}_1 \times R \rtimes S)\downarrow A \otimes O(R) \otimes \mathbf{R}O(C^*),$$

so $\mathcal{P}(A)$ is the full simplicial subcategory on fibrant cofibrant objects. The maps g_r give a morphism $g : P \to M$ in $\mathcal{M}(\mathbb{R}[t])$ and the maps ϕ_r give an isomorphism $\theta : M/tM \to E$ in $\mathcal{P}(\mathbb{R})$. We need to show that M is fibrant and cofibrant (so $M \in \mathcal{P}(\mathbb{R}[t])$) and that g is a quasi-isomorphism. Fibrancy is immediate, since the deformation of a surjection is a surjection.

Given an object $A \in \mathcal{M}(\mathbb{R}[t])$, the Mat_1-action gives a weight decomposition $A = \bigoplus_{n\geq 0} \mathcal{W}_n A$, and

$$A = \varprojlim_n{}^{\mathcal{M}(\mathbb{R}[t])} A/\mathcal{W}_n A.$$

Moreover, if $A \to B$ is a quasi-isomorphism, then so is $A/\mathcal{W}_n A \to B/\mathcal{W}_n B$ for all n. In order to show that M is cofibrant, take a trivial fibration $A \to B$ in $\mathcal{M}(\mathbb{R}[t])$ (i.e. a surjective quasi-isomorphism) and a map $M \to B$. Then $A/\mathcal{W}_n A \to B/\mathcal{W}_n B$ is a trivial fibration in $\mathcal{M}(\mathbb{R}[t])$, and in fact in $\mathcal{M}(\mathbb{R}[t]/t^n)$. Since $M_n \cong M/t^n M$ is cofibrant in $\mathcal{M}(\mathbb{R}[t]/t^n)$, the map $M \to B$ lifts to a map $M \to (A/\mathcal{W}_n A) \times_{B/\mathcal{W}_n B} B$. We now proceed inductively, noting that

$$(A/\mathcal{W}_{n+1} A) \times_{(B/\mathcal{W}_{n+1} B)} B \to (A/\mathcal{W}_n A) \times_{(B/\mathcal{W}_n B)} B$$

is a trivial fibration in $\mathcal{M}(\mathbb{R}[t]/t^{n+1})$. This gives us a compatible system of lifts $M \to (A/\mathcal{W}_n A) \times_{(B/\mathcal{W}_n B)} B$, and hence

$$M \to \varprojlim_n [(A/\mathcal{W}_n A) \times_{(B/\mathcal{W}_n B)} B] = A.$$

Therefore M is cofibrant.

To show that g is a quasi-isomorphism, observe that for $A \in \mathcal{M}(\mathbb{R}[t])$, the map $\mathcal{W}_n A \to \mathcal{W}_n(A/t^r A)$ is an isomorphism for $n < r$. Since g_r is a quasi-isomorphism for all r, this means that g induces quasi-isomorphisms $\mathcal{W}_n P \to \mathcal{W}_n M$ for all n, so g is a quasi-isomorphism. $\square$

Definition 5.15. Given an R-equivariant $O(R)$-augmented DGA $\mathscr{M}$ in the category of ind-MTS (resp. ind-MHS) of non-negative weights, define the associated non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $\mathbf{Spec}\,\zeta(\mathscr{M})$ as follows. Under Lemma 1.14 (resp. Lemma 1.8), the Rees module construction gives a flat $\mathrm{Mat}_1 \times R \times \mathbb{G}_m$-equivariant (resp. $\mathrm{Mat}_1 \times R \rtimes S$-equivariant) quasi-coherent $\mathscr{O}_{\mathbb{A}^1} \otimes O(R) \otimes \mathscr{O}_C$-augmented algebra $\xi(\mathscr{M})$ on $\mathbb{A}^1 \times C$ associated to $\mathscr{M}$. We therefore define $\mathbf{Spec}\,\zeta(\mathscr{M}) := \mathbf{Spec}_{\mathbb{A}^1\times C^*}\xi(\mathscr{M})|_{\mathbb{A}^1\times C^*}$.

Now, $\mathrm{gr}^W \mathscr{M}$ is an $O(R)$-augmented DGA in the category of Mat_1-representations (resp. $\bar{S}$-representations), so we may set $\underline{\mathrm{gr}}\mathbf{Spec}\,\zeta(\mathscr{M}) := \mathrm{Spec}\,\mathrm{gr}^W \mathscr{M}$. Since $\xi(\mathscr{M})$ is flat,

$$(\mathbf{Spec}\,\zeta(\mathscr{M})) \times^{\mathbf{R}}_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R} \simeq (\mathbf{Spec}\,\zeta(\mathscr{M})) \times_{\mathbb{A}^1,0} \mathrm{Spec}\,\mathbb{R},$$

so Lemma 1.14 (resp. Lemma 1.8) gives the required opposedness isomorphism.

Theorem 5.16. *For every non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) on a pointed Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$, there exists an R-equivariant $O(R)$-augmented DGA $\mathscr{M}$ in the category of ind-MTS (resp. ind-MHS) with $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) quasi-isomorphic in the category of algebraic mixed twistor (resp. mixed Hodge) structures to $\mathbf{Spec}\,\zeta(\mathscr{M})$, for ζ as above.*

Proof. Making use of Remark 5.10, choose a fibrant cofibrant replacement E for $O(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}})$ (resp. $O(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}})$) in the category $DG_{\mathbb{Z}}\mathrm{Alg}(R)_*(\mathrm{Mat}_1)$ (resp. $DG_{\mathbb{Z}}\mathrm{Alg}(R)_*(\bar{S})$), and a fibrant cofibrant replacement P for

$$\Gamma(C^*, \mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \otimes_{\mathscr{O}_{C^*}} \mathbf{R}\mathscr{O}_{C^*})$$
$$(\text{resp. } \Gamma(C^*, \mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}) \otimes_{\mathscr{O}_{C^*}} \mathbf{R}\mathscr{O}_{C^*}))$$

in the category

$$DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t]\otimes \mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)$$
$$(\text{resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t]\otimes \mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times S)).$$

Since P is cofibrant, it is flat, so the data of an algebraic mixed twistor (resp. mixed Hodge) structure give a quasi-isomorphism

$$f : P/tP \to E \otimes \mathbf{R}O(C^*)$$

in

$$DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)$$
$$(\text{resp. } DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times S)),$$

so we may apply Proposition 5.14 to obtain a fibrant cofibrant object

$$M \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t]\otimes \mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times \mathbb{G}_m)$$
$$(\text{resp. } M \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{R}[t]\otimes \mathbf{R}O(C^*)}(R)_*(\mathrm{Mat}_1 \times S))$$

with an isomorphism $M/tM \cong E \otimes \mathbf{R}O(C^*)$, and a quasi-isomorphism $g : P \to M$ lifting f.

Since M is cofibrant, it is flat as an $\mathbf{R}O(C^*)$-module. For the canonical map $\mathrm{row}_1^*\colon \mathbf{R}O(C^*) \to O(\mathrm{SL}_2)$, this implies that we have a short exact sequence

$$0 \to \mathrm{row}_1^* M(-1)\epsilon \to M \to \mathrm{row}_1^* M \to 0,$$

and the section $O(\mathrm{SL}_2) \to \mathbf{R}O(C^*)$ of graded rings (not respecting differentials) gives a canonical splitting of the short exact sequence for the underlying graded objects. Thus we may write $M^* = \mathrm{row}_1^* M \oplus \mathrm{row}_1^* M(-1)\epsilon$, and decompose the differential d_M as $d_M := \delta_M + N_M\epsilon$, where $\delta_M = \mathrm{row}_1^* d_M$.

Now, since $M/tM = E \otimes \mathbf{R}O(C^*)$, we know that

$$N_M\colon\ \mathrm{row}_{1*}\mathrm{row}_1^*(M/tM) \to \mathrm{row}_{1*}\mathrm{row}_1^*(M/tM)(-1)$$

is a surjection of sheaves on C^*. Since $M = \varprojlim_r M/t^r M$ is the Mat_1-equivariant category and M is flat, this means that N_M is also surjective. We therefore set

$$K := \ker(N_M\colon\ \mathrm{row}_{1*}\mathrm{row}_1^* M \to \mathrm{row}_{1*}\mathrm{row}_1^* M(-1));$$

as $\ker(N : \mathrm{row}_{1*}O(\mathrm{SL}_2) \to \mathrm{row}_{1*}O(\mathrm{SL}_2)(-1)) = \mathscr{O}_{C^*}$, we have

$$K \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1\times C^*}(\mathrm{Mat}_1 \times R \times \mathbb{G}_m)\downarrow O(\mathbb{A}^1 \times R) \otimes \mathscr{O}_{C^*}$$
$$(\text{resp. } K \in DG_{\mathbb{Z}}\mathrm{Alg}_{\mathbb{A}^1\times C^*}(\mathrm{Mat}_1 \times R \rtimes S)\downarrow O(\mathbb{A}^1 \times R) \otimes \mathscr{O}_{C^*}),$$

with

$$M = \Gamma(C^*, K \otimes_{\mathscr{O}_{C^*}} \mathbf{R}\mathscr{O}_{C^*}),$$

for $\mathbf{R}\mathscr{O}_{C^*}$ as in Definition 5.6.

Since M is flat over $\mathbf{R}O(C^*) \otimes O(\mathbb{A}^1)$, it follows that K is flat over $C^* \times \mathbb{A}^1$. Moreover, for $0 \in \mathbb{A}^1$, we have $0^*K = K/tK$, so

$$\begin{aligned} 0^*K &= \ker(N_M\colon \mathrm{row}_{1*}\mathrm{row}_1^*(M/tM) \to \mathrm{row}_{1*}\mathrm{row}_1^*(M/TM)(-1)) \\ &= E \otimes \ker(N : \mathrm{row}_{1*}O(\mathrm{SL}_2) \to \mathrm{row}_{1*}O(\mathrm{SL}_2)(-1)) \\ &= E \otimes \mathscr{O}_{C^*}. \end{aligned}$$

Thus K satisfies the opposedness condition, so by Lemma 1.14 (resp. Lemma 1.8) it corresponds to an ind-MTS (resp. ind-MHS) on the R-equivariant $O(R)$-augmented DGA algebra $(1,1)^*K$ given by pulling back along $(1,1)\colon \operatorname{Spec}\mathbb{R} \to \mathbb{A}^1 \times C$. Letting this ind-MTS (resp. ind-MHS) be $\mathscr{M}$ completes the proof. □

5.3.3. *Homotopy fibres.* In Proposition 5.14, it is natural to ask how unique the model M is. We cannot expect it to be unique up to isomorphism, but only up to quasi-isomorphism. As we will see in Corollary 5.20, that quasi-isomorphism is unique up to homotopy, which in turn is unique up to 2-homotopy, and so on.

Definition 5.17. Recall from [Ber] Theorem 1.1 that a morphism $F : \mathcal{C} \to \mathcal{D}$ in $s\mathrm{Cat}$ is said to be a weak equivalence (a.k.a. an ∞-equivalence) whenever

(W1) for any objects a_1 and a_2 in $\mathcal{C}$, the map $\underline{\mathrm{Hom}}_{\mathcal{C}}(a_1, a_2) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(Fa_1, Fa_2)$ is a weak equivalence of simplicial sets;

(W2) the induced functor $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

A morphism $F : \mathcal{C} \to \mathcal{D}$ in $s\mathrm{Cat}$ is said to be a fibration whenever

(F1) for any objects a_1 and a_2 in $\mathcal{C}$, the map $\underline{\mathrm{Hom}}_{\mathcal{C}}(a_1, a_2) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(Fa_1, Fa_2)$ is a fibration of simplicial sets;

(F2) for any objects $a_1 \in \mathcal{C}$, $b \in \mathcal{D}$, and homotopy equivalence $e : Fa_1 \to b$ in $\mathcal{D}$, there is an object $a_2 \in \mathcal{C}$ and a homotopy equivalence $d : a_1 \to a_2$ in $\mathcal{C}$ such that $Fd = e$.

Definition 5.18. Given functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ between categories, define the 2-fibre product $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C}$ as follows. Objects of $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C}$ are triples (a, θ, c), for $a \in \mathcal{A}, c \in \mathcal{C}$ and $\theta : Fa \to Gc$ an isomorphism in $\mathcal{B}$. A morphism in $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C}$ from (a, θ, c) to (a', θ', c') is a pair (f, g), where $f : a \to a'$ is a morphism in $\mathcal{A}$ and $g : c \to c'$ a morphism in $\mathcal{C}$, satisfying the condition that

$$Gg \circ \theta = \theta' \circ Ff.$$

Remark 5.19. This definition has the property that $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C}$ is a model for the 2-fibre product in the 2-category of categories. However, we will always use the notation $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C}$ to mean the specific model of Definition 5.18, and not merely any equivalent category.

Also note that

$$\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{C} = (\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{B}) \times_{\mathcal{B}} \mathcal{C},$$

and that a morphism $F : \mathcal{A} \to \mathcal{B}$ in $s\mathrm{Cat}$ is a 2-fibration in the sense of Definition 5.11 if and only if $\mathcal{A} \times^{(2)}_{\mathcal{B}} \mathcal{B} \to \mathcal{B}$ is a fibration in the sense of Definition 5.17.

Corollary 5.20. *Let $\mathcal{P} : \mathrm{Alg}(\mathrm{Mat}_1) \to s\mathrm{Cat}$ be one of the functors $\mathcal{PT}_*$ or $\mathcal{PH}_*$, and fix $E \in \mathcal{P}(\mathbb{R})$. Given an object E in $\mathcal{P}(\mathbb{R})$, the simplicial categories given by the homotopy fibre*

$$\mathcal{P}(\mathbb{R}[t]) \times^h_{\mathcal{P}(\mathbb{R})} \{E\}$$

and the 2-fibre

$$\mathcal{P}(\mathbb{R}[t]) \times^{(2)}_{\mathcal{P}(\mathbb{R})} \{E\}$$

are weakly equivalent.

Proof. By Proposition 5.13, $\mathcal{P}(\mathbb{R}[t]/t^r) \to \mathcal{P}(\mathbb{R})$ is a 2-fibration in $s\mathrm{Cat}$. Moreover, the proof of Proposition 5.14 shows that the map

$$\begin{aligned}\mathcal{P}(\mathbb{R}[t]) \to & \varprojlim_r{}^{(2)} \mathcal{P}(\mathbb{R}[t]/t^r) \\ & = \varprojlim_r [\mathcal{P}(\mathbb{R}[t]/t^r) \times^{(2)}_{\mathcal{P}(\mathbb{R}[t]/t^{r-1})} \mathcal{P}(\mathbb{R}[t]/t^{r-1}) \times^{(2)}_{\mathcal{P}(\mathbb{R}[t]/t^{r-2})} \cdots \times_{\mathcal{P}(\mathbb{R})} \mathcal{P}(\mathbb{R})]\end{aligned}$$

to the inverse 2-limit is an equivalence, so $\mathcal{P}(\mathbb{R}[t]) \to \mathcal{P}(\mathbb{R})$ is also a 2-fibration.

Therefore $\mathcal{P}(\mathbb{R}[t]) \times^{(2)}_{\mathcal{P}(\mathbb{R})} \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a fibration in the sense of Definition 5.17, so

$$\begin{aligned}\mathcal{P}(\mathbb{R}[t]) \times^h_{\mathcal{P}(\mathbb{R})} \{E\} &\simeq \mathcal{P}(\mathbb{R}[t]) \times^{(2)}_{\mathcal{P}(\mathbb{R})} \mathcal{P}(\mathbb{R}) \times_{\mathcal{P}(\mathbb{R})} \{E\} \\ &= \mathcal{P}(\mathbb{R}[t]) \times^{(2)}_{\mathcal{P}(\mathbb{R})} \{E\} \\ &= \mathcal{P}(\mathbb{R}[t]) \times^{(2)}_{\mathcal{P}(\mathbb{R})} \{E\},\end{aligned}$$

as required. □

5.3.4. SL_2*-splittings.*

Definition 5.21. An $\mathcal{S}$-splitting (or SL_2-splitting) of a mixed Hodge structure $(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MHS}}$ on a relative Malcev homotopy type is an isomorphism

$$\mathbb{A}^1 \times \underline{\mathrm{gr}}(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MHS}} \times \mathrm{SL}_2 \simeq \mathrm{row}_1^*(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MHS}},$$

in $\mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1\times\mathrm{SL}_2}(R)_*(\mathbb{G}_m \times S))$, giving row_1^* of the opposedness isomorphism on pulling back along $\{0\} \to \mathbb{A}^1$.

An $\mathcal{S}$-splitting (or SL_2-splitting) of a mixed twistor structure $(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MTS}}$ on a relative Malcev homotopy type is an isomorphism

$$\mathbb{A}^1 \times \underline{\mathrm{gr}}(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MTS}} \times \mathrm{SL}_2 \simeq \mathrm{row}_1^*(X,x)^{\rho,\mathrm{Mal}}_{\mathrm{MTS}},$$

in $\mathrm{Ho}(dg_{\mathbb{Z}}\mathrm{Aff}_{\mathbb{A}^1\times\mathrm{SL}_2}(R)_*(\mathbb{G}_m \times \mathbb{G}_m))$, giving row_1^* of the opposedness isomorphism on pulling back along $\{0\} \to \mathbb{A}^1$.

Corollary 5.22. *Every non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure* $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ *(resp.* $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$*) on a pointed Malcev homotopy type* $(X,x)^{R,\mathrm{Mal}}$ *admits a canonical* SL_2*-splitting.*

Proof. By Theorem 5.16, we have an R-equivariant $O(R)$-augmented DGA $\mathscr{M}$ in the category of ind-MTS (resp. ind-MHS) of non-negative weights, with $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) quasi-isomorphic in the category of algebraic mixed twistor (resp. mixed Hodge) structures to $\mathbf{Spec}\,\zeta(\mathscr{M})$.

By Theorem 4.15 (resp. Theorem 4.8) and Lemma 4.7, there is a unique $R \times \mathbb{G}_m$-equivariant (resp. $R \rtimes S$-equivariant) derivation $\beta\colon \mathrm{gr}^W\mathscr{M} \to (\mathrm{gr}^W\mathscr{M}) \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1)$, with the corresponding object

$$O(\mathbb{A}^1) \otimes (\mathrm{gr}^W\mathscr{M}) \otimes O(\mathrm{SL}_2) \xrightarrow{\beta+\mathrm{id}\otimes N} O(\mathbb{A}^1) \otimes (\mathrm{gr}^W\mathscr{M}, W) \otimes O(\mathrm{SL}_2)(-1)$$

isomorphic to the object M from the proof of Theorem 5.16 (with $\mathrm{gr}^W\mathscr{M}$ canonically isomorphic to E).

In particular, it gives a $\mathbb{G}_m \times R \times \mathbb{G}_m$-equivariant (resp. $\mathbb{G}_m \times R \rtimes S$-equivariant) isomorphism

$$\mathrm{row}_1^*\zeta(\mathscr{M}) \cong O(\mathbb{A}^1) \otimes (\mathrm{gr}^W\mathscr{M}) \otimes O(\mathrm{SL}_2).$$

Since $\mathbf{Spec}_{\mathbb{A}^1\times C^*}\mathscr{M}$ is by construction quasi-isomorphic to $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$), with $\mathrm{Spec}\,\mathrm{gr}^W\mathscr{M}$ quasi-isomorphic to $\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$), this gives us a quasi-isomorphism

$$\mathrm{row}_1^*\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \to \mathbb{A}^1\times \mathrm{Spec}\,(\mathrm{gr}^W\mathscr{M})\times \mathrm{SL}_2$$
$$(\text{resp. } \mathrm{row}_1^*\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \to \mathbb{A}^1\times \mathrm{Spec}\,(\mathrm{gr}^W\mathscr{M})\times \mathrm{SL}_2.)$$

□

Corollary 5.23. *If a pointed Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$ admits a mixed twistor structure $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$, then there is a canonical family*

$$\mathbb{A}^1\times (X,x)^{R,\mathrm{Mal}} \simeq \mathbb{A}^1\times \underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$$

of quasi-isomorphisms over $\mathbb{A}^1$.

Proof. Take the fibre of the SL_2-splitting

$$\mathrm{row}_1^*\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \simeq \mathbb{A}^1\times \underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}\times \mathrm{SL}_2$$

over $(1,1)\in\mathbb{A}^1\times C^*$. The fibre of $\mathrm{SL}_2\to C^*$ over 1 is $\left(\begin{smallmatrix}1 & 0\\ \mathbb{A}^1 & 0\end{smallmatrix}\right)$, giving the family of quasi-isomorphisms. □

5.3.5. *Homotopy groups.*

Corollary 5.24. *Given a non-positively weighted algebraic mixed twistor (resp. mixed Hodge) structure $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) on a pointed Malcev homotopy type $(X,x)^{R,\mathrm{Mal}}$, there are natural ind-MTS (resp. ind-MHS) on the the duals $(\varpi_n(X,x)^{\rho,\mathrm{Mal}})^\vee$ of the relative Malcev homotopy groups for $n\geq 2$, and on the Hopf algebra $O(\varpi_1(X,x)^{\rho,\mathrm{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n, with the Whitehead bracket and with the Hurewicz maps $\varpi_n(X^{\rho,\mathrm{Mal}})\to \mathrm{H}^n(X,\mathbb{O}(R))^\vee$ ($n\geq 2$) and $\mathrm{R_u}\varpi_1(X^{\rho,\mathrm{Mal}})\to \mathrm{H}^1(X,O(\mathbb{O}(R)))^\vee$, for $\mathbb{O}(R)$ as in Definition 1.18.

Proof. By Corollary 5.22, $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (resp. $(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}$) admits an SL_2-splitting. Therefore the conditions of [Pri4, Theorem 4.20] are satisfied, giving the required result. □

Note that Theorems 4.15 and 4.8 now show that the various homotopy groups have associated objects in STS or SHS, giving canonical SL_2-splittings. These splittings will automatically be the same as those constructed in [Pri4, Theorem 4.21] from the splitting on the homotopy type. Explicitly, they give canonical isomorphisms

$$(\varpi_n(X,x)^{R,\mathrm{Mal}})^\vee\otimes\mathcal{S}\cong(\mathrm{gr}^W\varpi_n(X,x)^{R,\mathrm{Mal}})^\vee\otimes\mathcal{S}$$

compatible with weight filtrations and with twistor or Hodge filtrations, and similarly for $O(\varpi_1(X,x)^{\rho,\mathrm{Mal}})$.

It is natural to ask whether the relative Malcev homotopy groups $\varpi_n(Y,y)^{Y,\mathrm{Mal}}$ are related to classical homotopy groups $\pi_n(Y,y)$. We now give conditions under which this is true.

Definition 5.25. Say that a group Γ is *n-good* with respect to a Zariski-dense representation $\rho\colon \Gamma\to R(k)$ to a reductive pro-algebraic group if for all finite-dimensional $\Gamma^{\rho,\mathrm{Mal}}$-representations V, the map

$$\mathrm{H}^i(\Gamma^{\rho,\mathrm{Mal}},V)\to\mathrm{H}^i(\Gamma,V)$$

is an isomorphism for all $i\leq n$ and an inclusion for $i=n+1$.

The following is [Pri5, Theorem 2.25], which strengthens [Pri2, Theorem 3.21]:

Theorem 5.26. *If (Y, y) is a pointed connected topological space with fundamental group Γ, equipped with a Zariski-dense representation $\rho\colon \Gamma \to R(\mathbb{R})$ to a reductive pro-algebraic groupoid for which:*

1. *Γ is $(N+1)$-good with respect to ρ,*
2. *$\pi_n(Y, y)$ is of finite rank for all $1 < n \leq N$, and*
3. *the Γ-representation $\pi_n(Y, y) \otimes_{\mathbb{Z}} \mathbb{R}$ is an extension of R-representations (i.e. a $\Gamma^{\rho,\mathrm{Mal}}$-representation) for all $1 < n \leq N$,*

then the canonical map

$$\pi_n(Y, y) \otimes_{\mathbb{Z}} \mathbb{R} \to \varpi_n(Y^{\rho,\mathrm{Mal}}, y)(\mathbb{R})$$

is an isomorphism for all $1 < n \leq N$.

To see how to compare homotopy groups when the goodness hypotheses are not satisfied, apply [Pri4, Theorem 3.10] to the universal cover of (Y, y).

5.4. **Quasi-projective varieties.** Fix a smooth projective complex variety X, a divisor D locally of normal crossings, and set $Y := X - D$. Let $j : Y \to X$ be the inclusion morphism. Take a Zariski-dense representation $\rho\colon \pi_1(Y, y) \to R(\mathbb{R})$, for R a reductive pro-algebraic group, with ρ having unitary monodromy around local components of D.

Definition 5.27. Define a functor G from DG algebras to pro-finite-dimensional chain Lie algebras as follows. First, write $\sigma A^\vee[1]$ for the brutal truncation (in non-negative degrees) of $A^\vee[1]$, and set

$$G(A) = \mathrm{Lie}(\sigma A^\vee[1]),$$

the free pro-finite-dimensional pro-nilpotent graded Lie algebra, with differential defined on generators by $d_A + \Delta$, with $\Delta\colon A^\vee \to (A \otimes A)^\vee$ here being the coproduct on $A^\vee$.

Given a DGA A with $A^0 = \mathbb{R}$, define

$$\pi_n(A) := \mathrm{H}_{n-1} G(A).$$

Corollary 5.28. *There are natural ind-MTS on the the duals $(\varpi_n(Y, y)^{\rho,\mathrm{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(Y, y)^{\rho,\mathrm{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n, with the Whitehead bracket and with the Hurewicz maps $\varpi_n(Y^{\rho,\mathrm{Mal}}) \to \mathrm{H}^n(Y, \mathbb{O}(R))^\vee$ $(n \geq 2)$ and $\mathrm{R_u}\varpi_1(Y^{\rho,\mathrm{Mal}}) \to \mathrm{H}^1(Y, O(\mathbb{O}(R)))^\vee$.

Moreover, there are canonical $\mathcal{S}$-linear isomorphisms

$$\begin{aligned}
\varpi_n(Y^{\rho,\mathrm{Mal},y})^\vee \otimes \mathcal{S} &\cong \pi_n(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_1)^\vee \otimes \mathcal{S} \\
O(\varpi_1(Y^{\rho,\mathrm{Mal}}, y)) \otimes \mathcal{S} &\cong O(R \ltimes \pi_1(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2)) \otimes \mathcal{S}
\end{aligned}$$

compatible with weight and twistor filtrations.

Proof. This just combines Theorem 3.19 (or Theorem 2.21 for a simpler proof whenever ρ has trivial monodromy around the divisor) with Corollary 5.24. The splitting comes from Corollary 5.22, making use of the isomorphism

$$\underline{\mathrm{gr}}\varpi_n(Y, y)^{R,\mathrm{Mal}}_{\mathrm{MTS}} = \mathrm{gr}^W \varpi_n(Y, y)^{R,\mathrm{Mal}}.$$

induced by the exact functor gr^W on MTS. □

Corollary 5.29. *If the local system on X associated to any R-representation underlies a polarisable variation of Hodge structure, then there are natural ind-MHS on the the duals $(\varpi_n(Y, y)^{\rho,\mathrm{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(Y, y)^{\rho,\mathrm{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n, with the Whitehead bracket and with the Hurewicz maps $\varpi_n(Y^{\rho,\mathrm{Mal}}) \to \mathrm{H}^n(Y, \mathbb{O}(R))^\vee$ $(n \geq 2)$ and $\mathrm{R_u}\varpi_1(Y^{\rho,\mathrm{Mal}}) \to \mathrm{H}^1(Y, O(\mathbb{O}(R)))^\vee$.

Moreover, there are canonical $\mathcal{S}$-linear isomorphisms

$$\begin{aligned} \varpi_n(Y^{\rho,\mathrm{Mal},y})^\vee \otimes \mathcal{S} &\cong \pi_n(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_1)^\vee \otimes \mathcal{S} \\ O(\varpi_1(Y^{\rho,\mathrm{Mal}}, y)) \otimes \mathcal{S} &\cong O(R \ltimes \pi_1(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2)) \otimes \mathcal{S} \end{aligned}$$

compatible with weight and Hodge filtrations.

Proof. This just combines Theorem 3.16 (or Theorem 2.22 for a simpler proof whenever ρ has trivial monodromy around the divisor) with Corollary 5.24, together with the splitting of Corollary 5.22. □

Proposition 5.30. *If the $(S^1)^\delta$-action on ${}^\nu\varpi_1(Y,y)^{\mathrm{red}}$ descends to R, then for all n, the map $\pi_n(Y,y) \times S^1 \to \varpi_n(Y^{\rho,\mathrm{Mal}}, y)_{\mathbb{T}}$, given by composing the map $\pi_n(Y,y) \to \varpi_n(Y^{\rho,\mathrm{Mal}}, y)$ with the $(S^1)^\delta$-action on $(Y^{\rho,\mathrm{Mal}}, y)_{\mathbb{T}}$ from Proposition 3.20, is continuous.*

Proof. The proof of [Pri4, Proposition 6.12] carries over to this generality. □

Corollary 5.31. *Assume that the $(S^1)^\delta$-action on ${}^\nu\varpi_1(Y,y)^{\mathrm{red}}$ descends to R, and that the group $\varpi_n(Y,y)^{\rho,\mathrm{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(Y,y)$. Then $\varpi_n(Y,y)^{\rho,\mathrm{Mal}}$ carries a natural $\mathcal{S}$-split mixed Hodge structure, which extends the mixed twistor structure of Corollary 5.28.*

Proof. The proof of [Pri4, Corollary 6.13] adapts directly. □

Remark 5.32. If we are willing to discard the Hodge or twistor structures, then Corollary 5.23 gives a family

$$\mathbb{A}^1 \times (Y^{\rho,\mathrm{Mal}}, y) \simeq \mathbb{A}^1 \times \mathrm{Spec}\,(\bigoplus_{a,b} \mathrm{H}^{a-b}(X, \mathbf{R}^b j_* \mathbb{O}(R))[-a], d_2)$$

of quasi-isomorphisms, and this copy of $\mathbb{A}^1$ corresponds to $\mathrm{Spec}\,\mathcal{S}$.

If we pull back along the morphism $\mathcal{S} \to \mathbb{C}$ given by $x \mapsto i$, the resulting complex quasi-isomorphism will preserve the Hodge filtration F (in the MHS case), but not $\bar{F}$. This splitting is denoted by b_i in Remark 4.11, and comparison with [Del4, Remark 1.3] shows that this is Deligne's functor a_F.

[Pri4, Propositionmhs-morganhodge] adapts to show that when $R = 1$, the mixed Hodge structure in Corollary 5.29 is the same as that of [Mor, Theorem 9.1]. Since a_F was the splitting employed in [Mor], we deduce that when $R = 1$, the complex quasi-isomorphism at $i \in \mathbb{A}^1$ (or equivalently at $\left(\begin{smallmatrix} 1 & 0 \\ i & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2$) is precisely the quasi-isomorphism of [Mor, Corollary 9.7].

Whenever the discrete S^1-action on $\varpi_n(Y,y)^{R,\mathrm{Mal}}_{\mathrm{MTS}}$ (from Proposition 3.20) is algebraic, it defines an algebraic mixed Hodge structure on $\varpi_n(Y,y)^{R,\mathrm{Mal}}$. In the projective case $(D = \emptyset)$, [KPT] constructed a discrete $\mathbb{C}^\times$-action on $\varpi_n(X,x)_{\mathbb{C}}$; via [Pri4, Remark 6.4], the comments above show that whenever the $\mathbb{C}^\times$-action is algebraic, it corresponds to the complex I^{pq} decomposition of the mixed Hodge structure, with $\lambda \in \mathbb{C}^\times$ acting on I^{pq} as multiplication by λ^p.

5.4.1. *Deformations of representations.* For $Y = X - D$ as above, and some real algebraic group G, take a reductive representation $\rho\colon \pi_1(Y,y) \to G(\mathbb{R})$, with ρ having unitary monodromy around local components of D. Write $\mathfrak{g}$ for the Lie algebra of G, and let $\mathrm{ad}\mathbb{B}_\rho$ be the local system of Lie algebras on Y corresponding to the adjoint representation $\mathrm{ad}\rho\colon \pi_1(Y,y) \to \mathrm{Aut}(\mathfrak{g})$.

Proposition 5.33. *The formal neighbourhood $\mathfrak{Def}_\rho$ of ρ in the moduli stack $[\mathrm{Hom}(\pi_1(Y,y),G)/G]$ of representations is given by the formal stack $[(Z,0)/\exp(\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho))]$, where $(Z,0)$ is the formal germ at 0 of the affine scheme Z given by*

$$\{(\omega,\eta)\in \mathrm{H}^1(X,j_*\mathrm{ad}\mathbb{B}_\rho)\oplus \mathrm{H}^0(X,\mathbf{R}^1j_*\mathrm{ad}\mathbb{B}_\rho)\,:\, d_2\eta+\frac{1}{2}[\omega,\omega]=0,\ [\omega,\eta]=0,\ [\eta,\eta]=0)\}.$$

The formal neighbourhood $\mathfrak{R}_\rho$ of ρ in the rigidified moduli space $\mathrm{Hom}(\pi_1(Y,y),G)$ of framed representations is given by the formal scheme

$$(Z,0)\times_{\exp(\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho))}\exp(\mathfrak{g}),$$

where $\exp(\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho))\subset\exp(\mathfrak{g})$ acts on $(Z,0)$ via the adjoint action.

Proof. Let R be the Zariski closure of ρ. This satisfies the conditions of Corollary 5.28, so we have an $\mathcal{S}$-linear isomorphism

$$O(\varpi_1(Y^{\rho,\mathrm{Mal}},y))\otimes\mathcal{S}\cong O(R\ltimes\pi_1(\bigoplus_{a,b}\mathrm{H}^{a-b}(X,\mathbf{R}^bj_*\mathbb{O}(R))[-a],d_2))\otimes\mathcal{S}$$

of Hopf algebras.

Pulling back along any real homomorphism $\mathcal{S}\to\mathbb{R}$ (such as $x\mapsto 0$) gives an isomorphism

$$\varpi_1(Y^{\rho,\mathrm{Mal}},y)\cong O(R\ltimes\pi_1(\bigoplus_{a,b}\mathrm{H}^{a-b}(X,\mathbf{R}^bj_*\mathbb{O}(R))[-a],d_2)).$$

We now proceed as in [Pri1, Remarks 6.6]. Given a real Artinian local ring $A=\mathbb{R}\oplus\mathfrak{m}(A)$, observe that

$$G(A)\times_{G(\mathbb{R})}R(\mathbb{R})\cong\exp(\mathfrak{g}\otimes\mathfrak{m}(A))\rtimes R(\mathbb{R}).$$

Since $\exp(\mathfrak{g}\otimes\mathfrak{m}(A))$ underlies a unipotent algebraic group, deformations of ρ correspond to algebraic group homomorphisms

$$\varpi_1(Y^{\rho,\mathrm{Mal}},y)\to\exp(\mathfrak{g}\otimes\mathfrak{m}(A))\rtimes R$$

over R.

Infinitesimal inner automorphisms are given by conjugation by $\exp(\mathfrak{g}\otimes\mathfrak{m}(A))$, and so [Pri2, Proposition 3.15] gives $\mathfrak{Def}_\rho(A)$ isomorphic to

$$[\mathrm{Hom}_R(\pi_1(\bigoplus_{a,b}\mathrm{H}^{a-b}(X,\mathbf{R}^bj_*\mathbb{O}(R))[-a],d_2),\exp(\mathfrak{g}\otimes\mathfrak{m}(A)))/\exp(\mathfrak{g}\otimes\mathfrak{m}(A))^R],$$

which is isomorphic to the groupoid of A-valued points of $[(Z,0)/\exp(\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho))]$.

The rigidified formal scheme $\mathfrak{R}_\rho$ is the groupoid fibre of $\mathfrak{Def}_\rho(A)\to B\exp(\mathfrak{g}\otimes\mathfrak{m}(A))$, which is just the set of A-valued points of $(Z,0)\times_{\exp(\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho))}\exp(\mathfrak{g})$, as in [Pri4, Proposition 3.25]. □

Remarks 5.34. The mixed twistor structure on $\varpi_1(Y^{\rho,\mathrm{Mal}},y)$ induces a weight filtration on the pro-Artinian ring representing $\mathfrak{R}_\rho$. Since the isomorphisms of Corollary 5.23 respect the weight filtration, the isomorphisms of Proposition 5.33 also do so. Explicitly, the ring $O(Z)$ has a weight filtration determined by setting $\mathrm{H}^{a-b}(X,\mathbf{R}^bj_*\mathbb{O}(R))$ to be of weight $a+b$, so generators of $O(Z)$ have weights -1 and -2. The weight filtration on the rest of the space is then characterised by the conditions that $\mathfrak{g}$ and $\mathrm{H}^0(Y,\mathrm{ad}\mathbb{B}_\rho)$ both be of weight 0.

Another interesting filtration is the pre-weight filtration J of Proposition 3.10. The constructions transfer this to a filtration on $\varpi_1(Y^{\rho,\mathrm{Mal}},y)$, and the $\mathcal{S}$-splittings (and hence Proposition 5.33) also respect J. The filtration J is determined by setting $\mathrm{H}^{a-b}(X,\mathbf{R}^bj_*\mathbb{O}(R))$ to be of weight b, so generators of $O(Z)$ have weights 0 and -1. We can then define $J_0Z:=\mathrm{Spec}\,O(Z)/J_{-1}O(Z)$, and obtain descriptions of $J_0\mathfrak{Def}_\rho\subset\mathfrak{Def}_\rho$

and $J_0\mathfrak{R}_\rho \subset \mathfrak{R}_\rho$ by replacing Z with J_0Z. These functors can be characterised as consisting of deformations for which the conjugacy classes of monodromy around the divisors remain unchanged — these are the functors studied in [Fot].

5.4.2. *Simplicial and singular varieties.* As in §3.4, let $X_\bullet$ be a simplicial smooth proper complex variety, and $D_\bullet \subset X_\bullet$ a simplicial divisor with normal crossings. Set $Y_\bullet = X_\bullet - D_\bullet$, assume that $|Y_\bullet|$ is connected, and pick a point $y \in |Y_\bullet|$. Let $j : |Y_\bullet| \to |X_\bullet|$ be the natural inclusion map.

Take $\rho\colon \pi_1(|Y_\bullet|, y) \to R(\mathbb{R})$ Zariski-dense, and assume that for every local system $\mathbb{V}$ on $|Y_\bullet|$ corresponding to an R-representation, the local system $a_0^{-1}\mathbb{V}$ on Y_0 is semisimple, with unitary monodromy around the local components of D_0.

Corollary 5.35. *There are natural ind-MTS on the the duals $(\varpi_n(|Y_\bullet|, y)^{\rho,\mathrm{Mal}})^\vee$ of the relative Malcev homotopy groups for $n \geq 2$, and on the Hopf algebra $O(\varpi_1(|Y_\bullet|, y)^{\rho,\mathrm{Mal}})$.*

These structures are compatible with the action of ϖ_1 on ϖ_n, with the Whitehead bracket and with the Hurewicz maps $\varpi_n(|Y_\bullet|^{\rho,\mathrm{Mal}}) \to \mathrm{H}^n(|Y_\bullet|, \mathbb{O}(R))^\vee$ $(n \geq 2)$ and $\mathrm{R_u}\varpi_1(|Y_\bullet|^{\rho,\mathrm{Mal}}) \to \mathrm{H}^1(|Y_\bullet|, O(\mathbb{O}(R)))^\vee$.

Moreover, there are canonical $\mathcal{S}$-linear isomorphisms

$$\begin{aligned}
\varpi_n(|Y_\bullet|^{\rho,\mathrm{Mal},y})^\vee \otimes \mathcal{S} &\cong \pi_n(\mathrm{Th}\,(\bigoplus_{p,q} \mathrm{H}^{p-q}(X_\bullet, \mathbf{R}^q j_* a^{-1}\mathbb{O}(R))[-p], d_1))^\vee \otimes \mathcal{S}\\
O(\varpi_1(|Y_\bullet|^{\rho,\mathrm{Mal}}, y)) \otimes \mathcal{S} &\cong O(R \ltimes \pi_1(\mathrm{Th}\,(\bigoplus_{p,q} \mathrm{H}^{p-q}(X_\bullet, \mathbf{R}^q j_* a^{-1}\mathbb{O}(R)[-p], d_1)))) \otimes \mathcal{S}
\end{aligned}$$

compatible with weight and twistor filtrations.

If $a_0^{-1}\mathbb{V}$ underlies a polarisable variation of Hodge structure on Y_0 for all $\mathbb{V}$ as above, then the ind-MTS above all become ind-MHS, with the $\mathcal{S}$-linear isomorphisms above compatible with Hodge filtrations.

Proof. The proofs of Corollaries 5.28 and 5.29 carry over, substituting Theorems 3.21 and 3.22 for Theorems 3.19 and 3.16. □

Corollary 5.36. *Assume that the $(S^1)^\delta$-action on ${}^\nu\varpi_1(Y_0, y)^{\mathrm{red}}$ descends to R, and that the group $\varpi_n(|Y_\bullet|, y)^{\rho,\mathrm{Mal}}$ is finite-dimensional and spanned by the image of $\pi_n(|Y_\bullet|, y)$. Then $\varpi_n(|Y_\bullet|, y)^{\rho,\mathrm{Mal}}$ carries a natural $\mathcal{S}$-split mixed Hodge structure, which extends the mixed twistor structure of Corollary 5.35.*

Proof. This is essentially the same as Corollary 5.31, replacing Proposition 3.19 with Proposition 3.25. □

Remark 5.37. When $R = 1$, [Pri4, Proposition 9.15] adapts to show that the mixed Hodge structure of Corollary 5.35 agrees with that of [Hai, Theorem 6.3.1].

5.4.3. *Projective varieties.* In [Pri4, Theorems 5.14 and 6.1], explicit SL_2 splittings were given for the mixed Hodge and mixed twistor structures on a connected compact Kähler manifold X. Since any MHS or MTS has many possible SL_2-splittings, it is natural to ask whether those of [Pri4] are the same as the canonical splittings of Corollary 5.22. Apparently miraculously, the answer is yes:

Theorem 5.38. *The quasi-isomorphisms*

$$\mathrm{row}_1^*(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}} \simeq \mathbb{A}^1 \times \mathrm{Spec}\,(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \times \mathrm{SL}_2$$
$$\text{and } \mathrm{row}_1^*(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}} \to \mathbb{A}^1 \times \mathrm{Spec}\,(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}}) \times \mathrm{SL}_2$$

of Corollary 5.22 are homotopic to the corresponding quasi-isomorphisms of [Pri4, Theorems 5.14 and 6.1].

Proof. Given a MTS or MHS V, an SL_2-splitting $\mathrm{row}_1^*\xi(V) \cong (\mathrm{gr}^W V) \otimes O(\mathrm{SL}_2)$ gives rise to a derivation $\beta\colon \mathrm{gr}^W V \to \mathrm{gr}^W V \otimes \Omega(\mathrm{SL}_2/C^*)$, given by differentiation with respect to $\mathrm{row}_1^*\xi(V)$. Since $\Omega(\mathrm{SL}_2/C^*) \cong O(\mathrm{SL}_2)(-1)$, this SL_2-splitting corresponds to the canonical SL_2-splitting of Theorem 4.15 or 4.8 if and only if $\beta(\mathrm{gr}^W V) \subset \mathrm{gr}^W V \otimes \mathrm{row}_2^\sharp O(C)(-1)$.

Now, the formality quasi-isomorphisms of [Pri4, Theorems 5.14 and 6.1] allow us to transfer the derivation $N\colon \mathrm{row}_1^*\mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}}) \to \mathrm{row}_1^*\mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}})(-1)$ to an N-linear derivation (determined up to homotopy)

$$N_\beta\colon\ E \otimes O(\mathrm{SL}_2) \to E \otimes O(\mathrm{SL}_2)(-1),$$

for any fibrant cofibrant replacement E for $O(\underline{\mathrm{gr}}(X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}})$, and similarly for $\mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}})$. Moreover, $\mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MTS}})$ (resp. $\mathscr{O}((X,x)^{R,\mathrm{Mal}}_{\mathrm{MHS}})$) is then quasi-isomorphic to the cone

$$\mathrm{row}_{1*}(E \otimes O(\mathrm{SL}_2) \xrightarrow{N_\beta} E \otimes O(\mathrm{SL}_2)(-1)).$$

If we write $N_\beta = \mathrm{id} \otimes N + \beta$, for $\beta\colon E \to E \otimes O(\mathrm{SL}_2)(-1)$, then the key observation to make is that the formality quasi-isomorphism coincides with the canonical quasi-isomorphism of Corollary 5.22 if and only if for some choice of β in the homotopy class, we have

$$\beta(E) \subset E \otimes \mathrm{row}_2^\sharp O(\mathbb{A}^2)(-1) \subset E \otimes O(\mathrm{SL}_2)(-1).$$

Now, [Pri4, Remark 4.22] characterises the homotopy class of derivations β in terms of minimal models, with $[\beta] = [\alpha + \gamma_x]$, where γ_x characterises the basepoint, and α determines the unpointed structure. In [Pri4, Theorem 8.13], the operators α and γ_x are computed explicitly in terms of standard operations on the de Rham complex.

For co-ordinates $\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right)$ on SL_2, it thus suffices to show that α and γ_x are polynomials in x and y. The explicit computation expresses these operators as expressions in $\tilde{D} = uD + vD^c$, $\tilde{D}^c = xD + yD^c$ and $h_i = G^2 D^* D^{c*} \tilde{D}^c$, where G is the Green's operator. However, each occurrence of $\tilde{D}$ is immediately preceded by either $\tilde{D}^c$ or by h_i. Since

$$\tilde{D}^c \tilde{D} = (xD + yD^c)(uD + vD^c) = (uy - vx) D^c D = D^c D,$$

we deduce that α and γ_x are indeed polynomials in x and y, so the formality quasi-isomorphisms of [Pri4, Theorems 5.14 and 6.1] are just the canonical splittings of Corollary 5.22. □

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